19 Fourier Series

This Maple V segment illustrates how Maple V can be utilized to study Fourier Series. Numerical and graphical methods for estimating the error of the approximation for truncating the series to n terms are suggested. An example of the Gibb's Phenomenom is discussed.

Suppose a function, F(x), is defined and integrable on an interval [-L,L]. Then F(x) has a Fourier Expansion given by the formula

$$\mathrm{F}(x) = \frac{1}{2} a_0 + \left(\sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\,\pi\,x}{L}\right) + b_n \sin\left(\frac{n\,\pi\,x}{L}\right)\right)\right)$$

where for n=0,1,2,...,

$$a_n = \frac{\int_{-L}^{L} \mathrm{F}(x) \cos\left(\frac{n\,\pi\,x}{L}\right) dx}{L}$$

and

$$b_n = \frac{\int_{-L}^{L} \mathrm{F}(x) \sin\left(\frac{n\,\pi\,x}{L}\right) dx}{L}$$

If the function F is even or odd then the series can be simplified. For example if F(x) is even, *i.e*, F(x) = F(-x) then the series reduces to the Cosine series

$$\mathrm{F}(x) = \frac{1}{2} a_0 + \left(\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\,\pi\,x}{L}\right)\right)$$

where

$$a_n = 2\,\frac{\int_{0}^{L} \mathrm{F}(x) \cos\left(\frac{n\,\pi\,x}{L}\right) dx}{L}$$

If F(x) is odd, *i.e.*, if F(-x) = -F(x), then the Fourier Series reduces to the Sine series

$$\mathrm{F}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\,\pi\,x}{L}\right)$$

where

$$b_n = 2\,\frac{\int_{0}^{L} \mathrm{F}(x) \sin\left(\frac{n\,\pi\,x}{L}\right) dx}{L}$$

Our first example, a "saw tooth" function, is the odd extension of the following function, resulting in a function of period 2 π.

$$F(x) = x, \quad \text{if} \quad 0 \le x <= \pi/2,$$

and

$$F(x) = \pi - x, \quad \text{for} \quad \pi/2 \le x \le \pi.$$

This function can be defined (at least for the interval $[-\pi, \pi]$) by means of the Heaviside Function. When using this function with Maple V it is convenient to abbreviate the built-in **Heaviside** Function with the **alias**, H.

```
> alias(H=Heaviside);
```

$$I, H$$

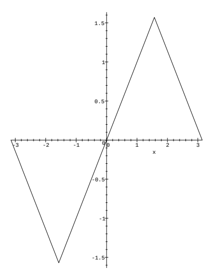
The desired function (in the interval $[-\pi, \pi]$) can now be defined.

```
> F :=x -> -(Pi+x)*(H(x+3*Pi/2)-H(x+Pi/2))+x*(H(x+Pi/2)-H(x-Pi/2))+
  (Pi-x)*H(x-Pi/2);
```

$$F := x \to -(\pi + x)\left(\mathrm{H}\left(x + \frac{3}{2}\pi\right) - \mathrm{H}\left(x + \frac{1}{2}\pi\right)\right) + x\left(\mathrm{H}\left(x + \frac{1}{2}\pi\right) - \mathrm{H}\left(x - \frac{1}{2}\pi\right)\right) + (\pi - x)\,\mathrm{H}\left(x - \frac{1}{2}\pi\right)$$

We now give a plot of the function F(x) on the interval $[-\pi, \pi]$. Note the use of the punctuation symbols!

```
> Plot1 := plot(F(x),x=-Pi..Pi): ";
```



Observe that since F(x) is odd that the Fourier expansion consists only of sines. Thus we need only compute, the b's. The appropriate Maple V command is

```
> b := n -> (2/Pi)*(int(x*sin(n*x),x=0..Pi/2) + int((Pi-x)*sin(n*x),
       x=Pi/2..Pi));
```

$$b := n \to 2\,\frac{\int_0^{1/2\,\pi} x \sin(n\,x)\,dx + \int_{1/2\,\pi}^{\pi} (\pi - x)\sin(n\,x)\,dx}{\pi}$$

A few values of b can be written to see if things look correct.

```
> b(1);b(2);b(3);b(4);b(5);
```

$$4\,\frac{1}{\pi}$$

$$0$$

$$-\frac{4}{9}\frac{1}{\pi}$$

$$0$$

$$\frac{4}{25}\frac{1}{\pi}$$

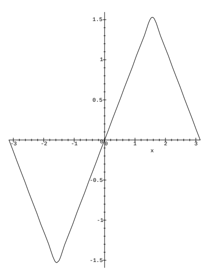
Now we can write the first n terms of the Fourier expansion.

```
> s := (n,x) -> sum(b(k)*sin(k*x),k=1..n);
```

$$s := (n, x) \rightarrow \sum_{k=1}^{n} \mathrm{b}(k)\sin(k\,x)$$

A plot of the first 15 terms of this expansion is given below.

```
plot(s(15,x),x=-Pi..Pi);
```

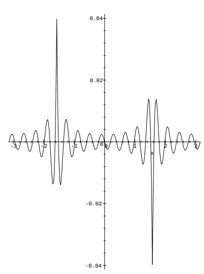


One way to see how well the first 15 terms of the Fourier expansion for F approximates F is to compute the corresponding error function or the difference between the approximation and the original function.

```
> e := (n,x)-> s(n,x) - F(x);
```

$$e := (n, x) \rightarrow \mathrm{s}(n, x) - \mathrm{F}(x)$$

```
> plot(e(15,x),x=-Pi..Pi);
```

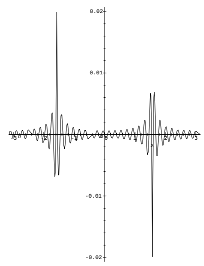


Note: At first glance the error seems to be large at $\pi/2$, but note that the value that it actually has is small. By using the mouse to move to near the point with x coordinate $pi/2$ at the bottom of the graph of e(15,x) and clicking once we can estimate that

$$|e(15, x)| < .04$$

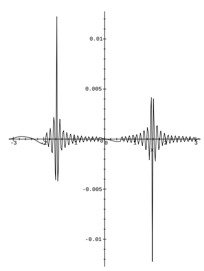
for all x in the interval $[-\pi, \pi]$.

```
> plot(e(31,x),x=-Pi..Pi);
```



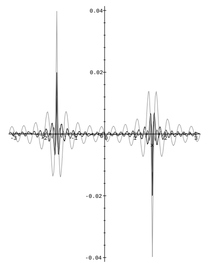
This time, using the mouse as above, we estimate the error to be bounded by .021.

```
> plot(e(51,x),x =-Pi..Pi);
```



This time, using the mouse as above, we estimate the error to be bounded by .015.

```
> plot({e(15,x),e(31,x),e(51,x)},x=-Pi..Pi);
```



By observing the error function for different values of n it seems safe to conjecture that the error in the approximation, e(n,x), becomes small uniformly in x as n gets large.

Animation Demonstration:

Let us now make an animation that provides graphical evidence of how well the finite sums approximates the function F(x). First we make a plot of the function F(x) to use later. The following loop develops six frames that show a plot of F(x) along with s(n,x) on the same graph for values of n equal to

$$1, 3, 5, \ldots, 11.$$

```
> i := 'i';
```

$$i := i$$

```
> for i from 0 to 5 do
> m := 2*i+1;
> R := plots[textplot]([-1.5,1,`n = `.m]):
> Q := plot(s(m,x),x=-Pi..Pi):
> P[i] := plots[display]({Plot1, R, Q}):
> od:
> i := 'i';
```

$$i := i$$

The next command creates an animation with 6 frames.

```
> plots[display]([seq(plot(P[i]),i=0..5)],insequence=true);
```

The output of the last Maple V instruction should be an animated sequence of graphs that compares the nth approximation, s(n,x), to F(x) for six different values of x.

The next set of animations compares the error made in each approximation with F(x).

```
> i := 'i';
```

$$i := i$$

```
> for i from 0 to 5 do
> m := 2*i+1;
> R := plots[textplot]([-1.5,1,'n = '.m]):
Q := plot(e(m,x),x=-Pi..Pi):
> P[i] := plots[display]({Plot1, R, Q}):
od:
> i := 'i';
```

$$i := i$$

```
> plots[display]([seq(P[i],i=0..5)],insequence=true);
```

The preceding Maple V output is a sequence of plots of the the error function, e(n,x), along with the original function F(x). It should give you you some idea of how well the nth approximation approximates F(x).

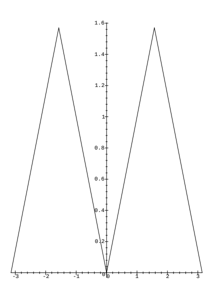
Let us now take the even periodic extension of the above function, F(x). We can again use the Heaviside Function to define this extension in the interval $[-\pi, \pi]$. We will call this function G(x).

```
> G := (x+Pi)*(H(x+Pi)-H(x+Pi/2)) - x*(H(x+Pi/2)-H(x))+
      x*(H(x)-H(x-Pi/2))+(Pi-x)*H(x-Pi/2);
```

$$G := (x+\pi)\left(\mathrm{H}(x+\pi) - \mathrm{H}\left(x+\frac{1}{2}\pi\right)\right) - x\left(\mathrm{H}\left(x+\frac{1}{2}\pi\right) - \mathrm{H}(x)\right) + x\left(\mathrm{H}(x) - \mathrm{H}\left(x-\frac{1}{2}\pi\right)\right) + (\pi - x)\,\mathrm{H}\left(x-\frac{1}{2}\pi\right)$$

We can plot the function just to make sure that it is the desired function.

```
> plot(G,x=-Pi..Pi);
```



Since G is an even function we must approximate it with a cosine series. First we compute the a's.

```
>  a :=n -> (2/Pi)*(int(x*cos(n*x),x=0..Pi/2)+int((Pi-x)*cos(n*x),
      x=Pi/2..Pi));
```

$$a := n \to 2\,\frac{\int_0^{1/2\,\pi} x\cos(n\,x)\,dx + \int_{1/2\,\pi}^{\pi} (\pi - x)\cos(n\,x)\,dx}{\pi}$$

Note that this is analogous to the formula for the b's except cosines replace sines. Evaluate a few of the terms:

```
> a(0);a(1);a(2);
```

$$\frac{1}{2}\,\pi$$

$$0$$

$$-2\,\frac{1}{\pi}$$

Now we can compute a Fourier Cosine Sum.

```
> s := (n,x) -> a(0)/2 + sum(a(k)*cos(k*x),k=1..n);
```

$$s := (n, x) \to \frac{1}{2}\,\mathrm{a}(0) + \left(\sum_{k=1}^{n} \mathrm{a}(k)\cos(k\,x)\right)$$

It is instructive to plot s(15,x) and G(x) on the same graph. One can make a similar analysis as above concerning estimating the error of approximation when replacing G(x) by s(n,x) for some values of n.

Sometimes there is a surprise in store for us as we try to obtain error estimates. In case the function x has a jump discontinuity we can observe an unusual phenomenom. In this example we consider the odd periodic extension of period $2\,\pi$ of the function:

$$F1(x) = \pi - x, \quad 0 \le x \le \pi.$$

Note: This function has jump discontinuities at

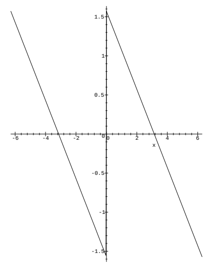
$$\ldots - \pi, 0, \pi, \ldots.$$

We can obtain, by using the Heaviside Function, a closed form for the function F1 in the interval $[-2\pi, 2\pi]$.

```
>  F1 := x -> (Pi-x)*H(x)/2-(Pi+x)*H(-x)/2;
```

$$F1 := x \to \frac{1}{2}\,(\pi - x)\,\mathrm{H}(x) - \frac{1}{2}\,(\pi + x)\,\mathrm{H}(-x)$$

```
> Plot1 :=
plot(F1(x),x=-2*Pi..2*Pi): ";
```



```
> b := n -> (2/Pi)*int((Pi-x)/2*sin(n*x),x=0..Pi);
```

$$b := n \to 2\,\frac{\displaystyle\int_0^{\pi} \frac{1}{2}\,(\pi - x)\sin(n\,x)\,dx}{\pi}$$

```
> b(1);b(2);b(3);
```

$$1$$

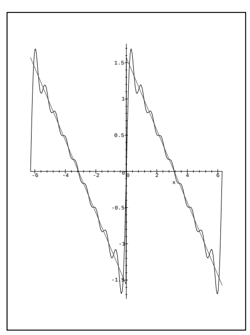
$$\frac{1}{2}$$

$$\frac{1}{3}$$

```
> s := (n,x) -> sum(b(k)*sin(k*x),k=1..n);
```

$$s := (n, x) \to \sum_{k=1}^{n} \mathrm{b}(k)\sin(k\,x)$$

```
> plot({s(9,x),F1(x)},x=-2*Pi..2*Pi);
```

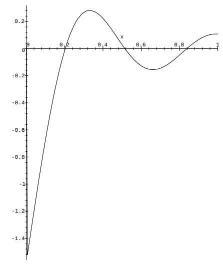


In this case there is a noticeable “hump” in the curve of s(9,x) near the discontinuities at 0, -2 π, and 2 π. To analyze this situation let us examine the error function.

```
> e := (n,x) -> s(n,x) - F1(x);
```

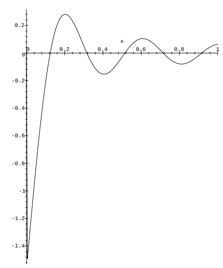
$$e := (n, x) \rightarrow \mathrm{s}(n, x) - \mathrm{F1}(x)$$

```
> plot(e(9,x),x=0..1);
```

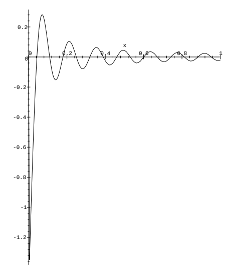


We see that the error term takes on a value nearly equal to $-\pi/2$ at $x = 0$. This is not surprising since, at the discontinuity, theory suggests that the Fourier Series converges to the value $(F(0_-) + F(0_+))/2$ which is 0 in this case since $F1(0_-) = -\pi/2$ and $F1(0_+) = \pi/2$. The thing to observe here is the size of the "hump" which is located near .31. Here we see the error is nearly equal to .28.

```
> plot(e(15,x),x=0..1);
```

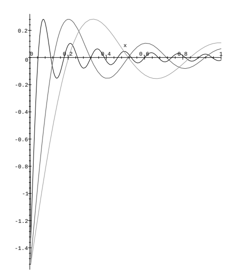


```
> plot(e(44,x),x=0..1);
```



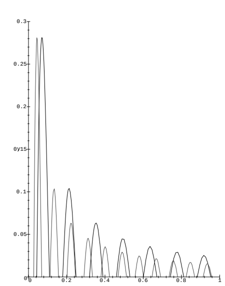
The “hump” of approximately the same size is still present even for this much larger value of n.

```
> plot({e(9,x),e(15,x),e(44,x)},x=0..1);
```



Note that each of the curves e(9,x),e(15,x), and e(44,x) oscillate and each have a maximum value of about .28. It seems that this maximum value is nearly .28 no matter what value n assumes. For example:

```
> plot({e(44,x),e(70,x)},x=0..1,y=0..(0.3));
```



This as an example of what is known as Gibb’s Phenomenom and it can be shown that the actual value of the maximum approaches the value of the integral:

$$\int_0^\pi \frac{\sin(z)}{z}\,dz - \frac{1}{2}\pi$$

```
> evalf(int(sin(z)/z,z=0..Pi))-evalf(Pi/2);
```

$$.281140725$$

Thus it can be proven (This is not easy.) that the maximum peak of e(n,x) is around .28 for all n.

Animation Demonstration:

Let us now make an animation that gives further graphical amplification to these ideas on the Gibb’s Phenomenom. As before we develop six frames each of which is a plot of the finite sum, s(n,x), along with F(x) for values of $n = 1, 5, 9, \ldots, 21$.

```
> i := ’i’;
```

$$i := i$$

```
> for i from 0 to 5 do
> m := 4*i+1;
> R := plots[textplot]([-3.0,1,‘n = ‘.m]):
> Q := plot(s(m,x),x=-2*Pi..2*Pi):
> P[i] := plots[display]({Plot1, R, Q}):
od:
> i := ’i’;
```

$$i := i$$

```
> plots[display]([seq(P[i],i=0..5)],insequence=true);
```

We finish the animation analysis with the following Maple V segment, which produces six frames of the error function , e(n,x), along with the function F(x), for the values $n = 1, 5, 9, \ldots, 21$.

```
> i := ’i’;
```

$$i := i$$

```
> for i from 0 to 5 do
> m := 4*i+1;
> R := plots[textplot]([-3.0,1,‘n = ‘.m]):
```

```
> Q := plot(e(m,x),x=-2*Pi..2*Pi):
```

```
> P[i] := plots[display]({Plot1, R, Q}):
```

```
> od:
```

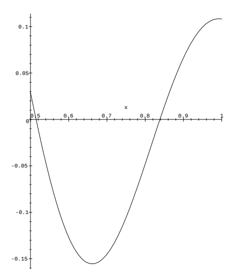
```
> i := 'i';
```

$$i := i$$

```
> plots[display]([seq(P[i],i=0..5)],insequence=true);
```

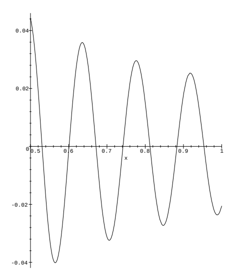
If we stay away from points of discontinuity then convergence is again uniform. The following sequence of commands give an indication that the error gets smaller with large n over the interval [0.5,1].

```
> plot(e(9,x),x=.5..1);
```



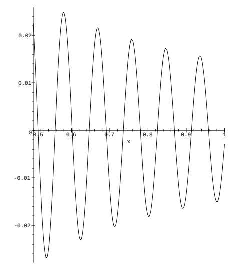
In This case —e(9,x)— takes on a maximum value of around 0.156.

```
> plot(e(44,x),x=.5..1);
```



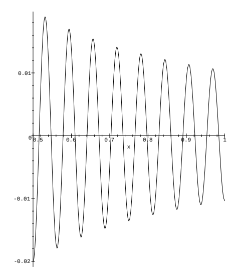
Here the error seems to be bounded by about 0.043.

```
plot(e(70,x),x=.5..1);
```



Here the maximum error is bounded by 0.027. One might feel that there could be a “hump” that is simply smaller than the one found above. But if we plot more points then the error continues to diminish. If we taken = 100 then

```
plot(e(100,x),x=.5..1);
```



The error in this case seems bounded by 0.02.

Exercises 19.0

1. Using the Heaviside Function write a Maple V statement that yields the even 2π periodic extension of the function:

$$G1(x) = \pi - x, \quad 0 \leq x \leq \pi$$

valid over the interval $[-2\pi, 2\pi]$. **Note:** This means that your plot will extend over two periods of the function.

2. Write a Maple V function that defines the “a” coefficients in the Fourier Cosine expansion of G1(x).

3. Write a Maple V function that defines the nth partial sum, s(n,x), of the Fourier expansion.

4. Plot s(3,x), s(9,x), s(15,x) and G1(x) on the same graph in the interval $[-\pi, \pi]$ and paste the plot into your worksheet.

5. Write a Maple V function, e(n,x), that computes the error in approximating G1(x) by s(n,x).

6. Plot the function e(3,x) on the interval $[-\pi, \pi]$.

7. Using the preceding plot estimate the maximum error in approximating the function G1 by s(3,x) on the interval $[-\pi, \pi]$.

8. The plot of the function e(9,x) on the interval $[-\pi, \pi]$.

9. Using the plot estimate the maximum error in approximating the function G1 by s(9,x) on the interval $[-\pi, \pi]$.

10. Using graphical methods find the smallest value of n that will give a maximum error of less than 0.04 when approximating G1(x) by s(n,x). Justify your answer with plots.