18 Solving Systems of Linear Differential Equations

This Chapter illustrates how Maple V's built-in linear algebra package can be useful in solving linear systems with constant coefficients.

We will use routines that come from the Maple V Library **linalg**. This library can be used in a session by invoking the command:

```
> with(linalg):

Warning: new definition for   norm

Warning: new definition for   trace
```

Note: In the above statement we used **:** instead of **;** to suppress output. The first example involves solving the homogeneous linear system

$$X' = AX,$$

where A is the 3 by 3 matrix:

$$A := \begin{bmatrix} -1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix}$$

The way to solve such a system involves the following steps:

1. Find the eigenvalues and eigenvectors.
2. Write the general solution.

The first step can sometimes be very laborious to solve by hand even for systems as small as 3 by 3. Maple can be very useful in solving these problems. We now enter matrix.

```
> A := matrix(3,3,[-1,1,2,1,0,1,2,1,-1]);
```

$$A := \begin{bmatrix} -1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix}$$

The command **eigenvects** gives the eigenvalues and corresponding eigenvectors in one step. If you enter the following command then Maple will return the eigenvalues, that except for their order should be as follows:

```
> eigsA := eigenvects(A);
```

$$eigsA := [-1, 1, \{[1\ -2\ 1]\}], [-3, 1, \{[-1\ 0\ 1]\}], [2, 1, \{[1\ 1\ 1]\}]$$

The last expresson is a sequence of lists the first element of eigsA, eigsA[1], is the list

$$[-3, 1, [-1, 0, 1]].$$

The first element of this list is given by eigsA[1][1] is -3 and is equal to one of the eigenvalues. The second element of this list,eigsA[1][2], is the order of the eigenvalue (in this case 1). The final element in this list, eigsA[1][3], is the set of eigenvectors corresponding to the eigenvalue, eigsA[1][1]. In this case there is only one eigenvector,

$$\text{eigsA[1][3][1]} = [-1, 0, 1].$$

Note: The order of the elements of eigsA vary from one time or another and thus sometime it might turn out that eigsA[1] is

$$[-1, 1, [1, -2, 1]].$$

In another Maple seesion the last Maple V command might lead to a different order. Once we have eigenvalues and enough independent eigenvectors to span the eigenspace of the A we can write the general solution.

```
> solA := sum(c[i]*exp(eigsA[i][1]*t)*eigsA[i][3][1],i=1..3);
```

$$solA := c_1\,e^{(-t)}\,[1\;-2\;1] + c_2\,e^{(-3t)}\,[-1\;0\;1] + c_3\,e^{(2t)}\,[1\;1\;1]$$

```
> solA := evalm(solA);
```

$$solA := \Big[\; c_1\,e^{(-t)} - c_2\,e^{(-3t)} + c_3\,e^{(2t)} \quad -2\,c_1\,e^{(-t)} + c_3\,e^{(2t)} \quad c_1\,e^{(-t)} + c_2\,e^{(-3t)} + c_3\,e^{(2t)} \;\Big]$$

The Maple command, **map**, allows to verify that solA is indeed a solution. Substituting X = solA into X' = A X first involves differentiating solA.

```
> map(diff,solA,t);
```

$$\Big[- c_1\,e^{(-t)} + 3\,c_2\,e^{(-3t)} + 2\,c_3\,e^{(2t)} \quad 2\,c_1\,e^{(-t)} + 2\,c_3\,e^{(2t)} \quad -c_1\,e^{(-t)} - 3\,c_2\,e^{(-3t)} + 2\,c_3\,e^{(2t)}\Big]$$

The last result should equal A X.

```
> evalm(A &* solA);
```

$$\Big[- c_1\,e^{(-t)} + 3\,c_2\,e^{(-3t)} + 2\,c_3\,e^{(2t)} \quad 2\,c_1\,e^{(-t)} + 2\,c_3\,e^{(2t)} \quad -c_1\,e^{(-t)} - 3\,c_2\,e^{(-3t)} + 2\,c_3\,e^{(2t)}\Big]$$

Thus solA is a solution. Since all of the eigenvalues are distinct we have an eigenspace with the right dimension and the solution is a general solution.

In applications we usually are required to solve initial value problems. The next thing we do is find the solution of X'= A X that satisfies the initial conditions:

$$x_1(0) = 0, \quad x_2(0) = -2, \quad x_3(0) = 1.$$

First we solve for the constants $c_1,\ c_2,\ c_3$.

```
> sol := simplify(solve({subs(t=0,solA[1])=0,subs(t=0,solA[2])=-2,
> subs(t=0,solA[3])=1},{c[1],c[2],c[3]}));
```

$$sol := \left\{c_2 = \frac{1}{2}, c_3 = \frac{-1}{3}, c_1 = \frac{5}{6}\right\}$$

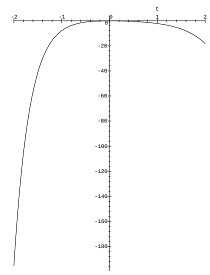
We now substitute the values of c_1, c_2, and c_3 into the general solution to obtain the desired particular solution:

```
> solAa := [subs(sol,solA[1]),subs(sol,solA[2]),subs(sol,solA[3])];
```

$$solAa := \left[\frac{5}{6}\,e^{(-t)} - \frac{1}{2}\,e^{(-3t)} - \frac{1}{3}\,e^{(2t)}, -\frac{5}{3}\,e^{(-t)} - \frac{1}{3}\,e^{(2t)}, \frac{5}{6}\,e^{(-t)} + \frac{1}{2}\,e^{(-3t)} - \frac{1}{3}\,e^{(2t)}\right]$$

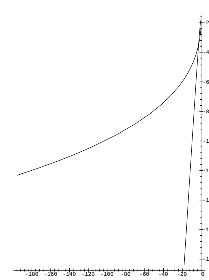
Once we know a particular solution we can ask for graphical information about the solution.For example a plot of x_1 versus time can be given by

```
> plot(solAa[1],t=-2..2,title = ‘x1 vs t‘);
```



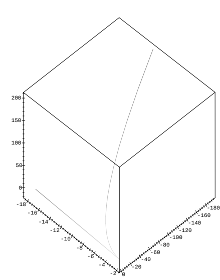
We can use a parametric plot to plot x_1 vs x_2 or x_1 vs x_3 or any other pair of dependent variables.

```
> plot([solAa[1],solAa[2],t=-2..2],title = ‘x1 vs x2‘);
```



Since the solution curve lies in three dimensional space it is also useful to plot it using the Maple function **spacecurve** which is contained in the package **plots**. Thus we invoke **plots**.

```
> with(plots):
> spacecurve([solAa[1],solAa[2],solAa[3]],t=-2..2,axes = BOXED);
```



Of course **dsolve** can be used to find both general and particular solutions:

```
> eq1 := D(x1)(t) =A[1,1]* x1(t)+A[1,2]*x2(t)+A[1,3]*x3(t);
```

$$eq1 := \mathrm{D}(x1)(t) = -\mathrm{x1}(t) + \mathrm{x2}(t) + 2\,\mathrm{x3}(t)$$

```
> eq2 := D(x2)(t) =A[2,1]* x1(t) + A[2,2]*x2(t) +A[2,3]* x3(t);
```

$$eq2 := \mathrm{D}(x2)(t) = \mathrm{x1}(t) + \mathrm{x3}(t)$$

```
> eq3 := D(x3)(t) = A[3,1]*x1(t) + A[3,2]*x2(t) +A[3,3]* x3(t);
```

$$eq3 := \mathrm{D}(x3)(t) = 2\,\mathrm{x1}(t) + \mathrm{x2}(t) - \mathrm{x3}(t)$$

```
> dsolve({eq1,eq2,eq3},{x1(t),x2(t),x3(t)});
```

$$\{\mathrm{x2}(t) = \_C1\,\mathrm{e}^{(2t)} - 2\,\_C3\,\mathrm{e}^{(-t)}, \mathrm{x3}(t) = \_C1\,\mathrm{e}^{(2t)} + \_C2\,\mathrm{e}^{(-3t)} + \_C3\,\mathrm{e}^{(-t)},$$
$$\mathrm{x1}(t) = \_C1\,\mathrm{e}^{(2t)} - \_C2\,\mathrm{e}^{(-3t)} + \_C3\,\mathrm{e}^{(-t)}\}$$

```
> dsolve({eq1,eq2,eq3,x1(0)=0,x2(0)=-2,x3(0)=1},{x1(t),x2(t),x3(t)});
```

$$\left\{\mathrm{x2}(t) = -\frac{5}{3}\,\mathrm{e}^{(-t)} - \frac{1}{3}\,\mathrm{e}^{(2t)}, \mathrm{x1}(t) = \frac{5}{6}\,\mathrm{e}^{(-t)} - \frac{1}{2}\,\mathrm{e}^{(-3t)} - \frac{1}{3}\,\mathrm{e}^{(2t)},\right.$$
$$\left.\mathrm{x3}(t) = \frac{5}{6}\,\mathrm{e}^{(-t)} + \frac{1}{2}\,\mathrm{e}^{(-3t)} - \frac{1}{3}\,\mathrm{e}^{(2t)}\right\}$$

The question that you should ask is: does this solution agree with the previous solution? Be careful, notice that the last solution is a set of equations and their order is unimportant.

If the eigenvalues are complex then the procedure is a little more complicated. The next example shows how to find the general solution of

$$X' = BX$$

where

$$B := \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

First enter this matrix. **Note:** There are two different techniques for entering a 3 by 3 matrix into a Maple session. Compare the command

```
> B := matrix([[1,0,0],[2,1,-2],[3,2,1]]);
```

we are about to use for entering B and the command

```
> A := matrix(3,3,[1,1,2,1,2,1,2,1,1]);
```

that we used for entering A. It seems that Maple treats both methods equally.

```
> B := matrix([[1,0,0],[2,1,-2],[3,2,1]]);
```

$$B := \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

As before, the solution of such a system requires that we

1. Find the eigenvalues and eigenvectors.
2. Write the general solution.

Again we use the Maple command **eigenvects**.

```
> eigsB :=
```

```
> eigenvects(B);
```

$$eigsB := \left[1, 1, \left\{\left[1 \; \frac{-3}{2} \; 1\right]\right\}\right],$$
$$\left[\text{RootOf}(\_Z^2 - 2\_Z + 5), 1, \left\{\left[0 \; \frac{1}{2}\text{RootOf}(\_Z^2 - 2\_Z + 5) - \frac{1}{2} \; 1\right]\right\}\right]$$

The answer uses the notation

$$\text{RootOf}(\_Z^2 - 2\_Z + 5)$$

for the complex eigenvalues in this case. We will show how to deal with this in a moment, but first we will write the solution corresponding to the real eigenvalue 1.

(**Note:** Remember that an item listed in a set is unordered. Hence the real eigenvalue may appear as eigsB[j][1] for j= 1, 2, or 3. This particular time it occurs as the first element.)

```
> Sol[1]:= exp(eigsB[1][1]*t)*eigsB[1][3][1];
```

$$Sol_1 := e^t \left[1 \; \frac{-3}{2} \; 1\right]$$

```
> Sol[1] := evalm(");
```

$$Sol_1 := \left[e^t \; -\frac{3}{2}e^t \; e^t\right]$$

Turning to the complex eigenvalues, we will use the command allvalues.

```
> complexeig := allvalues(eigsB[2][1]);
```

$$complexeig := 1 + 2I, 1 - 2I$$

Substitute one component of the result of **allvalues** for the **RootOf** expression.

```
> comvect := subs(eigsB[2][1]=complexeig[1],eigsB[2][3]);
```

$$comvect := \{[\,0\; I\; 1\,]\}$$

```
> comsol := evalm(exp(complexeig[1]*t)*comvect[1]);
```

$$comsol := [\,0\; I\, e^{((1+2\,I)\,t)}\; e^{((1+2\,I)\,t)}\,]$$

Use **Re** and **Im** to obtain two real vector value solutions.

```
> Sol[2] := Re(comsol);
```

$$Sol_2 := \Re(\,comsol\,)$$

```
> Sol[3] := Im(comsol);
```

$$Sol_3 := \Im(\,comsol\,)$$

```
> evalm(Sol[2]);
```

$$[\,0\;\; -\Im(\,e^{((1+2\,I)\,t)}\,)\;\Re(\,e^{((1+2\,I)\,t)}\,)\,]$$

Construct the general solution and use **evalm** and **evalc** to evaluate the answer.

```
> solB := evalm(sum(c[i]*Sol[i],i=1..3));
```

$$solB := \Big[c_1\, e^t \;\; -\frac{3}{2}\,c_1\, e^t - c_2\,\Im(\,e^{((1+2\,I)\,t)}\,) + c_3\,\Re(\,e^{((1+2\,I)\,t)}\,)$$
$$c_1\, e^t + c_2\,\Re(\,e^{((1+2\,I)\,t)}\,) + c_3\,\Im(\,e^{((1+2\,I)\,t)}\,)\Big]$$

This does not look like a correct answer since Maple does not automatically simplify complex numbers. Thus we must apply the Maple command **evalc**.

```
> solB := map(evalc,solB);
```

$$solB := \Big[c_1\, e^t \;\; -\frac{3}{2}\,c_1\, e^t - c_2\, e^t \sin(\,2\,t\,) + c_3\, e^t \cos(\,2\,t\,)$$
$$c_1\, e^t + c_2\, e^t \cos(\,2\,t\,) + c_3\, e^t \sin(\,2\,t\,)\Big]$$

We can check to see if the equations are satisfied.

```
> map(diff,solB,t);
```

$$\Big[c_1\, e^t \;\; -\frac{3}{2}\,c_1\, e^t - c_2\, e^t \sin(\,2\,t\,) - 2\,c_2\, e^t \cos(\,2\,t\,) + c_3\, e^t \cos(\,2\,t\,) - 2\,c_3\, e^t \sin(\,2\,t\,)$$
$$c_1\, e^t + c_2\, e^t \cos(\,2\,t\,) - 2\,c_2\, e^t \sin(\,2\,t\,) + c_3\, e^t \sin(\,2\,t\,) + 2\,c_3\, e^t \cos(\,2\,t\,)\Big]$$

```
> evalm(B &* solB);
```

$$\left[c_1\,e^t - \frac{3}{2}c_1\,e^t - c_2\,e^t\sin(2\,t) - 2\,c_2\,e^t\cos(2\,t) + c_3\,e^t\cos(2\,t) - 2\,c_3\,e^t\sin(2\,t)\right.$$
$$\left. c_1\,e^t + c_2\,e^t\cos(2\,t) - 2\,c_2\,e^t\sin(2\,t) + c_3\,e^t\sin(2\,t) + 2\,c_3\,e^t\cos(2\,t)\right]$$

Let us now find a solution to an initial value problem and plot it.

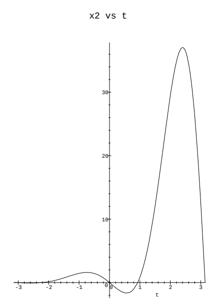
```
> sol2 := simplify(solve({subs(t=0,solB[1])=
-1,subs(t=0,solB[2])=0,subs(t=0,solB[3])=1},{c[1],c[2],c[3]}));
```

$$sol2 := \left\{c_1 = -1,\, c_2 = 2,\, c_3 = \frac{-3}{2}\right\}$$

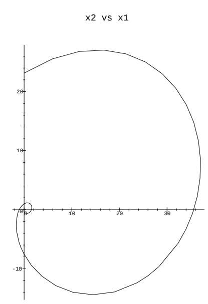
```
> solBa := [subs(sol2,solB[1]),subs(sol2,solB[2]),subs(sol2,solB[3])];
```

$$solBa := \left[-e^t,\, \frac{3}{2}e^t - 2\,e^t\sin(2\,t) - \frac{3}{2}e^t\cos(2\,t),\, -e^t + 2\,e^t\cos(2\,t) - \frac{3}{2}e^t\sin(2\,t)\right]$$

```
> plot(solBa[2],t=-Pi..Pi,title = ‘x2 vs t‘);
```

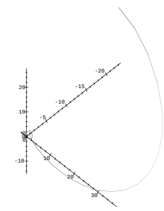



```
> plot([solBa[2],solBa[3],t=-Pi..Pi],  title = ‘x2 vs x1‘);
```




```
> with(plots):
```

```
> spacecurve([solBa[1],solBa[2],solBa[3]],t=-Pi..Pi,axes = NORMAL);
```



As before we can solve the differential equation directly using **dsolve**. First we enter the equation in the proper form.

```
> eqB1 :=diff(x1(t),t) = B[1,1]*x1(t)+B[1,2]*x2(t) +B[1,3]*x3(t);
```

$$eqB1 := \frac{\partial}{\partial t}\,\mathrm{x1}(t) = \mathrm{x1}(t)$$

```
> eqB2 := diff(x2(t),t) = B[2,1]*x1(t)+B[2,2]*x2(t)+B[2,3]*x3(t);
```

$$eqB2 := \frac{\partial}{\partial t}\,\mathrm{x2}(t) = 2\,\mathrm{x1}(t) + \mathrm{x2}(t) - 2\,\mathrm{x3}(t)$$

```
> eqB3 := diff(x3(t),t) = B[3,1]*x1(t) + B[3,2]*x2(t)+B[3,3]*x3(t);
```

$$eqB3 := \frac{\partial}{\partial t}\,\mathrm{x3}(t) = 3\,\mathrm{x1}(t) + 2\,\mathrm{x2}(t) + \mathrm{x3}(t)$$

```
> dsolve({eqB1,eqB2,eqB3},{x1(t),x2(t),x3(t)});
```

$$\Big\{\mathrm{x3}(t) = \_C1\,\mathrm{e}^t + \_C2\,\mathrm{e}^t\sin(2\,t) + \_C3\,\mathrm{e}^t\cos(2\,t),$$
$$\mathrm{x2}(t) = -\frac{3}{2}\,\_C1\,\mathrm{e}^t - \_C3\,\mathrm{e}^t\sin(2\,t) + \_C2\,\mathrm{e}^t\cos(2\,t),\ \mathrm{x1}(t) = \_C1\,\mathrm{e}^t\Big\}$$

```
> dsolve({eqB1,eqB2,eqB3,x1(0)=-1,x2(0)=0,x3(0)=1},{x1(t),x2(t),x3(t)});
```

$$\Big\{\mathrm{x1}(t) = -\mathrm{e}^t,\ \mathrm{x3}(t) = -\mathrm{e}^t - \frac{3}{2}\,\mathrm{e}^t\sin(2\,t) + 2\,\mathrm{e}^t\cos(2\,t),$$
$$\mathrm{x2}(t) = \frac{3}{2}\,\mathrm{e}^t - 2\,\mathrm{e}^t\sin(2\,t) - \frac{3}{2}\,\mathrm{e}^t\cos(2\,t)\Big\}$$

Exercises 18.0

1. Enter the matrix

$$A := \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ -8 & -5 & -2 \end{bmatrix}$$

into a Maple V worksheet.

(a) Use the Maple V command **eigenvects** to find the eigenvalues and corresponding eigenvectors of A.

(b) Write the general solution of the differential equation $X' = AX$.

2. Enter the matrix

$$B := \begin{bmatrix} -3 & 2 & 3 \\ 0 & -2 & 2 \\ 0 & -2 & -3 \end{bmatrix}$$

into a Maple V worksheet.

(a) Use the Maple command **eigenvects**,and **allvalues** to find the eigenvalues and corresponding real and complex eigenvectors of B.

(b) Write the general solution of the differential equation $X' = BX$ in real form.