Some MAPLE exercises for M302 - Differential Equations

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Contents

Chapter 1

Introduction

1.1 General Caveat

This book contains the handouts for *M302 - Ordinary differential equations* that I developed in 1995 (spring and Fall). The following should be noted:

1. These handouts are MAPLE examples. No "group project" or other impressive piece of curriculum development is here (unfortunately)!

2. The handouts are "rough" and accompanied by considerable discussion in the classroom (which does not appear in these notes). Some were done piece-by-piece during the class, others done in a single sitting during class and still others were take home exercises.

3. This was my first year teaching differential equations with MAPLE, and my first use of MAPLE. So I have probably missed the point in a number of areas, and may not have used the most efficient MAPLE commands in all cases.

4. Not recorded in these notes are various demonstrations I gave to the students during the course of the semester (mainly in MATLAB).

1.2 Sources and Acknowledgments

Much of the material in these notes are "computerizations" of material in the text-book *Elementary Differential Equations and Boundary Value Problems* by Boyce and DiPrima (fifth edition). Frequently the examples are taken directly from the book and reference is made to examples in the book. This book is the course book for M302.

The other main source of material is the *Differential Equations Laboratory Notebook* by Borrelli, Coleman and Boyce. Unfortunately this notebook is not oriented to MAPLE. Many of the best examples in these notes are MAPLEizations of work in the notebook.

Chapter 2

Unix Account Setup

Welcome to class MATH302-013!

During this semester, you will be using central UNIX systems to complete class assignments. I recommend you use the X-terminal sites around campus since they operate similarly (but not exactly the same) as the Sun workstations in our classroom. However, you only NEED to use an X-terminal if you want to obtain graphics on the screen.

2.1 Activating your UD computer account

If you have used one of the computers named Brahms, Chopin, or Strauss in the past, then your *username* and *password* are what they were before. If you have never used a central UNIX system, you can get your account activated by taking your student ID card (M–F, 8am–5pm) to a *Password Changing Station* in the Student Services Building or in 002A Smith Hall.

If your userid is still the original 5-digit number, you should select and change it to a username (e.g., marydoe, dprince) by typing the `username` command. Details on these operations are in the UNIX Handbook, available at the University Bookstore, and in the UNIX Hands-on Worksheet available in the CNS Reference Library (014 Smith).

2.2 Setting up your UD account

This section shows how to set up your account for this class. This needs to be done regardless of whether you are using an X-terminal, dumb terminal or your own PC to log in to the public computers. *Make sure that you type all commands exactly as shown.*

- **Log on to a campus UNIX machine:** Exactly how to do this depends on whether you are using a PC, terminal, X-terminal etc.;

- **Set up access to commands:** *You will only need to do this step once, not every time you log in.* Enter the following command *exactly* as you see it:

  `~monk/Class/MATH302XXX/student-setup`

  where `XXX` is replaced by 013. This has the effect of making sure that you will have proper access to files, commands, etc. for this class.

2.3 Working on MATH302-013class assignments outside the Math. Lab.

1. **To begin:** type `newgrp 2077`. The group/project number for your project is **2077** (if you forget this number, you can see all your projects by typing `chdgrp`).

2. **Do your work.** The files related to this class for your instructor's account are in the directory $CLASSHOME. For example, if you are told to print your instructor's file named *instructions*, you should refer to it by the full name *$CLASSHOME/instructions*. This only works after you have typed the `newgrp 2077` command (i.e. you have done step 1 above).

3. **End your work** for this class: type `exit`. Then, you can either log out or work on something else.

If at any time you need to make sure that 2077 is the current group, type `id` and the number which will appear in parenthesis following `gid` will be the current group. If you get any messages asking for a password or other messages, you are not in the project (maybe you just added the class?). See me.

You will automatically see announcements your instructor posts. To review the contents of files in the MATH302-013's *motd* ("message of the day") directory, use the following commands, called *aliases*.

What you type	What the command displays
classinfo	The contents of the most recently changed message file in the *motd* directory
classinfo *filename*	The contents of the file named *filename* in the *motd* directory
listclassinfo	A directory listing of the files in the *motd* directory
allclassinfo	The filename and first 24 lines of every file in the *motd* directory

For example, typing `classinfo exam1-change` will display the contents of *monk/Class/MATH302013/motd/exam1-change*.

2.4 Using Maple from X-terminals

2.4.1 Activating your computer account

If you have used one of the computers named Brahms, Chopin, or Strauss, in the past, then your *username* and *password* are what they were before. If you have not, then take your student ID (M-F 8am-5pm) to a *Password Changing Station* in the Student Services Building or in 002A Smith Hall. There activate your computer account and set a password.

X-terminal sites

It is best to do your Maple work on an X-terminal. Some of the public X-terminal sites are

009B Willard Hall - 23 X-terminals; M-R 8am-10pm, F 8am-5pm, Sat 12noon-5pm, Sun 12noon-5pm.
114 Newark/Pearson Hall - 24 X-terminals; M-R 9am-midnight, F 9am-5pm, Sat 12noon-4pm, Sun 6pm-midnight.
002 Smith Hall - 28 X-terminals; daily 8am-midnight

2.4.2 Setting up the M302 environment

The following must be done *only once* this semester. At one of the X-terminal sites

- login using your username/usernumber and password.
- move mouse to blue part of the screen and click once the *right* mouse button. In menu which pops up, *left* click one of the lines which says *Brahms, Chopin*, or *Strauss*.
- in new window which pops up, type *carefully*

  ```
  ~monk/Class/MATH302013/student-setup
  ```

  (Don't forget the ~) and hit *Return*. The environment is setup.
- move mouse to blue part and *right* click once. In the menu, *left* click the line *Quit and Logout*.

2.4.3 Starting Maple

You should have completed the section *Setting up the M302 environment*.

- login using your username/usernumber and password.
- move mouse to blue part of screen and *right* click the mouse button. In menu which pops up, *left* click one of the lines which says *Brahms, Chopin*, or *Strauss*.
- in the new window which pops up, type *carefully*

  ```
  newgrp 2077  Hit Return
  xmaple &     Hit Return
  ```

Maple should start up - be patient. If you quit Maple accidentally and want to restart it; just type `xmaple &` in the same window and Maple will restart.

2.4.4 Logging out from an X-terminal

Be sure to save your Maple work (as described in the Tutorial) if you will need it later. Next, move mouse to blue part and *right* click once. In the menu, *left* click the line *Quit and Logout. If you don't do this someone else could wipe out your work.*

Chapter 3

Integral Curves and Direction Fields

3.1 Introduction

The purpose of this programing assignment is two fold. First we will set up your account to make sure that you can

1. Access Maple
2. Print graphs and text on the printer in the computer lab.

Second we will do a short computer exercise which will show you how to draw integral curves and direction fields.

3.2 Getting started in the math lab

If the screen on the workstation is blank, move the mouse on the mouse pad to light up the screen. You should see a standard X-windows login screen. If you don't see the prompt, ask the instructor.

Log on to a lab SUN UNIX machine Login as if you were logging into one of the composers (with your usual user-name and password). Then wait. The window system will start up and the screen will turn blue (this takes about a minute). If you don't manage to login, ask the instructor.

Run MAPLE Now you are ready to use MAPLE. Just move the mouse until the pointer is in the blue background. Then hold down the left mouse button to obtain the applications menu. With the button held down, move the pointer until the MAPLE entry is high-lighted then let go the button. A window should appear announcing MAPLE's appearance. Move the pointer into the MAPLE window and you are ready to start. Commands for MAPLE are typed directly into this window.

Log-out from the machine You must do this after you have finished the assignment otherwise strangers may be able to access your account and ruin your work. But don't logout until you have finished the assignment! To logout, simply move the pointer to the blue background and (using the mouse). Then hold down the right mouse button to activate the *root menu*. Move the pointer down to the *Exit Window System and logout* entry and release the mouse button. You will be asked to confirm exiting "mwm". Confirm this. The screen should change to show the "Solaris" sunburst.

3.3 Drawing integral curves.

We shall start by drawing the integral curves like those shown in Figure 2.1.1 on page 17 in your book (this is so you can check that everything worked right). In your book, the general solution to the ODE

$$y' = -\frac{1}{2}y + \frac{3}{2}$$

is shown to be

$$y = 3 + c\exp(-x/2)$$

To draw integral curves of this equation, start MAPLE. The first step is to tell MAPLE to use the differential equation package and then define the right hand side and differential equation:

```
with(DEtools);
f:=-y/2+3/2;
ode:=diff(y(x),x)=f;
```

Next we give Maple some starting points. These are points on the integral curves you want to draw (this is a trial and error problem to get the best choice). For this example, a reasonable choice is (type this into MAPLE)

```
inits:={[-1,-0.5],[0,0],[0,2],[-1,4],[0,5]};
```

Finally we can draw the integral curves (notice that you don't have to solve the differential equation, just tell MAPLE the differential equation):

```
DEplot(ode,[x,y],x=-1..5,inits);
```

Now try to draw some integral curves for the problem from class

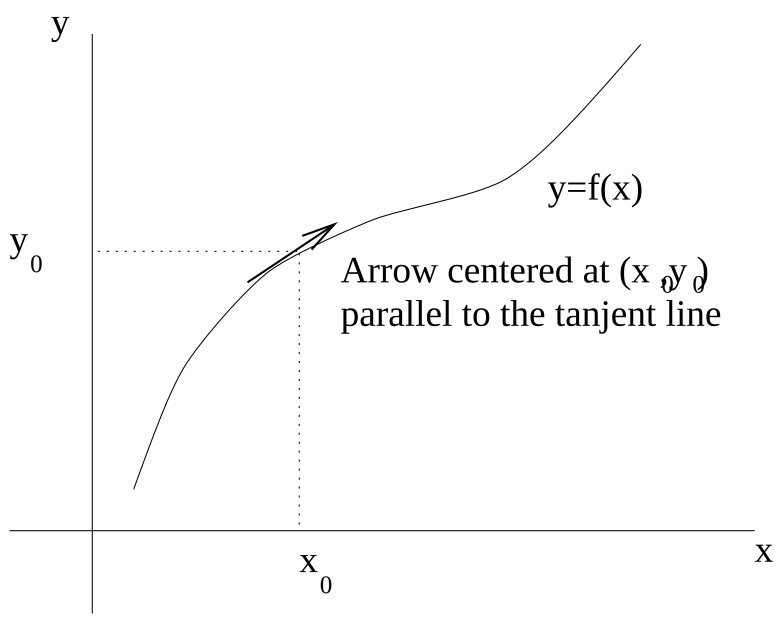
$$y' = -3y$$

Here the general solution is

$$y = c\exp(-3x)$$

Choose limits and initial values to obtain a clear picture of how the solutions behave.

Figure 3.1: A schematic illustrating the construction of one arrow of the direction field.



3.4 Drawing direction fields.

Let the rate function $f(x, y)$ in the first order ODE $y' = f(x, y)$ be defined in a rectangle R in the $x - y$ plane given by $a \le x \le b$ and $c \le y \le d$. A **solution curve** of this ordinary differential equation is a function $y(x)$ whose graph is in R and for which $y'(x) = f(x, y(x))$ for $a \le x \le b$. From calculus we know that if (x_0, y_0) is a point on the solution curve $y = y(x)$ then $y'(x_0)$ is the slope of tangent line to $y = y(x)$ at $x = x_0$. We can represent this graphically by drawing a short line segment centered at (x_0, y_0) with slope $y'(x_0) = f(x_0, y_0)$. We can repeat this process at a number of grid points throughout R and obtain the **direction field** of the differential equation. If the grid is fine enough, the resulting direction field may "suggest" solution curves in R with the property that the tangent line to the curve at each point lies along the direction field at that point.[1]

To draw the direction field for the differential equation in the previous example

$$y' = -\frac{1}{2}y + \frac{3}{2}$$

you can use the MAPLE `dfieldplot` routine:

```
with(plots):
```

[1]This description is taken almost verbatim from *Differential Equations Laboratory Notebook* by Borrelli, Coleman and Boyce.

```
with(DEtools):
f:=-y/2+3/2;
ode:=diff(y(x),x)=f;
dfieldplot(ode,[x,y],x=-1..5,y=-2..6);
```

Better still would be to display the direction fields and the integral curves on one plot:

```
inits:={[-1,-0.5],[0,0],[0,2],[-1,4],[0,5]};
p1:=DEplot(ode,[x,y],x=-1..5,inits):
p2:=dfieldplot(ode,[x,y],x=-1..5,y=-2..6):
display({p1,p2});
```

An alternative way to get the direction field and integral curves on one plot is to use the MAPLE `DEplot1` command. Note that this command uses `f` rather than `ode` as an argument.

```
DEplot1(f,[x,y],x=-1..5,inits,y=-2..6);
```

Now consider the ordinary differential equation

$$y' + \frac{2}{x}y = 4x$$

Draw the direction field for this equation using the range $-3 \leq x \leq 3$ and $-1 \leq y \leq 5$. Plot also the integral curves through $[1, 2]$, $[1, 1]$, and $[-1, 1]$.

Now print the resulting graph by selecting *Print/Postscript* from the *File* menu on the Maple graph tool. You will then need to open a command tool window and print the resulting file using *lpr*.

What do you think happens when $x = 0$ for the differential equation?

It turns out that the general solution to the above ODE is

$$y(x) = x^2 + \frac{c}{x^2}$$

so the solution through $(1, 2)$ has $c = 1$ (ie the solution is $y(x) = x^2 + 1/x^2$) and through $(1, 1)$ is $y(x) = x^2$.

Compare your picture to Figure 2.2.1 of your book.

Congratulations! The first day is over. **Don't forget to logout!**

Chapter 4

Solving ODE's in MAPLE

4.1 Solving ODEs in MAPLE exactly

We shall start by solving the example on page 17 in your book (this is so you can check that everything worked right). In your book, the general solution to the ODE

$$y' = -\frac{1}{2}y + \frac{3}{2}$$

is shown to be

$$y = 3 + c\exp(-x/2)$$

To obtain this solution in MAPLE, start up MAPLE. Then load the Differential Equations MAPLE library and the plotting library as follows:

```
with(DEtools):
with(plots):
```

Next enter the ODE as before

```
f:=-y/2+3/2;
ode:=diff(y(x),x)=f;
```

Before solving the differential equation you can, if you wish, tell MAPLE the initial condition. For example here we shall use the initial condition

$$y(0) = y_0$$

so we enter

```
ic:=y(0)=y0;
```

Now use the MAPLE command `dsolve` to find the general solution

```
soln:=dsolve({ode,ic},y(x));
```

Compare the above solution to the one given above. Note that in this case MAPLE gives you an explicit solution $y(x) = \cdots$.

You can now plot integral curves using this solution by choosing different values of y_0 as follows: first you get the right hand side of the equation, then you plot it

```
phi:=rhs(soln);
plot({subs(y0=0,phi),subs(y0=-1,phi),subs(y0=1,phi)},x=0..5);
```

You can plot direction fields and integral curves as before (note the use of a colon (:) after the statement `p1:=...` which avoids a great deal of useless output:

```
p1:=plot({subs(y0=0,phi),subs(y0=-1,phi),subs(y0=1,phi)},x=0..5):
p2:=dfieldplot(f,[x,y],x=0..5,y=-1..3):
display({p1,p2});
```

4.2 Solving ODEs implicitly and drawing integral curves

Solutions to separable equations are often implicit functions of the form $g(x,y) = c$ for some constant c. So the MAPLE `dsolve` program will often return an implicit solution. For example consider the separable ODE

$$y\frac{dy}{dx} + (1+y^2)\sin(x) = 0.$$

which has the general solution

$$\frac{1}{2}\ln(1+y^2) - \cos(x) = c.$$

We can find the general solution in MAPLE as follows

As usual we start by loading `DEtools` and `plots` and enter the differential equation into MAPLE

```
with(DEtools):with(plots):
f:=-(1+y^2)*sin(x);
ode:=y*diff(y(x),x)=f;
soln:=dsolve(ode,y(x));
```

The solution returned is

$$soln := \mathrm{y}(\,x\,)^2 = -1 + \mathrm{e}^{(\,2\cos(\,x\,)\,)}\,\_C1$$

This is a simple implicit solution of the differential equation (a little different to the one I gave in class).

To plot integral curves, you need to substitute values for `_C1` and use the MAPLE program `implicitplot` to plot the solution. For example as follows (this may take a little time, so we will save the `implicitplot` output for later - note the colon at the end of the statement `p1:=...`)

```
p1:=implicitplot({subs(_C1=2,soln),subs(_C1=1,soln),subs(_C1=.2,soln),
subs(_C1=4,soln),subs(_C1=8,soln),subs(_C1=16,soln)},x=-1..5,y=-1..5):
display(p1);
```

The results of `implicitplot` can be used with `dfieldplot` to create a direction field and integral curve plot as follows (note the division by y since `dfieldplot` assumes $y' = f(x,y)$!):

```
p2:=dfieldplot(f/y,[x,y],x=-1..5,y=-1..5):
display({p1,p2});
```

4.3 Approximate solutions via `dsolve`

Last time we saw that MAPLE can use `dsolve` to solve ordinary differential equations. However in many cases `dsolve` will not be able to solve a differential equation exactly. So you will have to ask MAPLE to generate an approximate solution for you. In MAPLE this is also done using `dsolve` but with the `type=numeric` option. Suppose you want to solve the ordinary differential equation

$$y' = -2\frac{y}{x} + 4x$$

with the initial condition $y(0.1) = 4$.

The first step is to create a solution function as follows: we define the ordinary differential equation and the initial conditions that use `dsolve` with the `numeric` option (note that $y(x)$ is used).

```
ode:=diff(y(x),x)+2*y(x)/x-4*x=0;
ic:=y(0.1)=4;
soln:=dsolve({ode,ic},y(x),type=numeric);
```

Now you can use `soln` to evaluate a numerical approximation to $y(x)$ for different values of x:

```
soln(1);
soln(7);
```

If you wish to graph the solution, use `odeplot` from the `plots` package:

```
with(plots):
odeplot(soln,[x,y(x)],0.1..6);
```

Chapter 5

First Assignment (for credit)

This assignment is to be handed in on Friday 09/22.

5.1 Population models.

Logistic Equation. The logistic model of population says that if $p(t)$ is the population at time t, then

$$p' = kp - rp^2$$

Use the values (chosen entirely at random) of $k = 2$ and $r = 1$. Draw the direction field for this equation for $0 \leq x \leq 7$ and $0 \leq y \leq 4$. Then superimpose on this direction field the solutions computed by MAPLE using the initial data

$$p(0) = .1$$

and

$$p(0) = 4$$

From the direction field plot, if $p(0) > 0$, what is

$$\lim_{t\to\infty} p(t) =?$$

Spruce Budworm Model. The following is a model for the population of spruce bud-worm (and important factor in the timber industry I believe).

$$p' = p\left(k - bp - \frac{cp}{L + p^2}\right)$$

There are now negative terms: the logistic term $-bp^2$ which represents resource depletion, and the term $-cp^2/(L + p^2)$ which represents losses due to predators.

Use the parameter values:

b	.2	c	5
k	2	L	2

Draw a direction field plot for the model using $0 \le x \le 5$ and $0 \le y \le 8$. Superimpose the integral curves for $p(0) = .5$, $p(0) = 2$, $p(0) = 4$ and $p(0) = 8$. From the graph, what is the value of

$$\lim_{x \to \infty} p(x) = ?$$

when $p(0) > 0$. Does the limit depend on the value of $p(0)$? Have MAPLE draw a graph of

$$y = p\left(k - bp - \frac{cp}{L + p^2}\right)$$

against p for $0 \le p \le 8$ to help you understand the behavior of the ODE.

5.2 The Falling Raindrop

This example is from *Differential Equations Laboratory Notebook* by Borrelli, Coleman and Boyce. Consider a raindrop falling towards the earth near the earth's surface along the local vertical. Neglecting wind, the raindrop will be acted on by air resistance and gravitational forces. If the velocity of the raindrop is $v(t)$ at time t then v satisfies the differential equation

$$\begin{aligned} v' &= -g - \frac{k}{m} v|v| \\ v(0) &= v_0. \end{aligned}$$

Here $g = 9.8\ \mathrm{m/s^2}$ is the force due to gravity on a unit mass. m is the mass of the raindrop in kilograms and $k > 0$ is the drag coefficient. The initial velocity is v_0 and the term $v|v|$ arises since the force due to drag is proportional to v^2 and always acts to oppose motion (whatever the sign of v).

The first step in analyzing an initial value problem of this type is to "non-dimensionalize" by changing to dimensionless quantities. This has the advantage of removing lots of coefficients from the equation, and allows us to work with quantities that have magnitudes independent of the particular choice of coefficients.

In this example, if we define s to be dimensionless time then it proves convenient to define s by

$$t = \left(\frac{m}{kg}\right)^{1/2} s$$

and dimensionless velocity is denoted w and given by

$$v = \left(\frac{mg}{k}\right)^{1/2} w.$$

Change variables in the equation (ie substitute for v and t using w and s) and show that the differential equation becomes

$$\frac{dw}{ds} = -1 - w|w|$$

(Show your working in detail on a piece of paper!).

Draw a direction field plot for $0 \leq s \leq 5$ and $-2 \leq w \leq 3$ and draw integral curves for $w_0 = -2$, $w_0 = -1.5$, $w_0 = -1$, $w_0 = .2$, $w_0 = 0$, $w_0 = 1$, $w_0 = 2$ and $w_0 = 3$ (hand in a hard-copy of the plot and the MAPLE session used to generate it).

What is the value of

$$\lim_{s\to\infty} w(s) = ?$$

regardless of w_0? What does this say about

$$\lim_{t\to\infty} v(t) = ?$$

Chapter 6

Homogeneous second order ordinary differential equations

6.1 Linearity and independence of solutions of homogeneous problems

6.1.1 Superposition principle

In this section we shall use MAPLE to verify that a linear combination of solutions of a homogeneous linear differential equation is a solution of the differential equation. Consider the differential equations

$$x^2y'' - 2xy' + 2y = 0$$

First we shall verify that the functions $y_1(x) = x$ and $y_2(x) = x^2$ are solutions of this quation. We enter the differential equation into MAPLE and then substitute the two solutions into the equation:

```
ode:=x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x);
y1:=x;
simplify(subs(y(x)=y1,ode));
y2:=x^2;
simplify(subs(y(x)=y2,ode));
```

Next we take a general linear combination of y_1 and y_2 and verify it satisfies the equation:

```
y3:=c1*y1+c2*y2;
simplify(subs(y(x)=y3,ode));
```

As you can see the general linear combination satisfies the ODE.

If the differential equation is non-linear, it is no longer true in general that a linear combination of two solutions satisfies the differential equation. For example consider the differential equation

$$y'' + (y')^2 = 0$$

(this example is from *Differentail equations with applications and historical notes* by G.F. Simmons). We first verify as before that $y_1(x) = 1$ and $y_2(x) = \ln(x)$ are solutions of the equation:

```
restart;
ode:=diff(y(x),x$2)+(diff(y(x),x))^2;
y1:=1;
y2:=ln(x);
simplify(subs(y(x)=y1,ode));
simplify(subs(y(x)=y2,ode));
```

Next we substitute a general linear combination of these solutions into the equation and verify that in general the linear combination is not a solution of the differential equation

```
y3:=c1*y1+c2*y2;
simplify(subs(y(x)=y3,ode));
```

So *linearity is essential* for the superposition principle.

6.1.2 Linear independence

MAPLE can help you to decide if two solutions of a second order problem are linearly independent since MAPLE can compute the Wronskian for you. Suppose you want to verify that $y_1(x) = \exp(-2x)$ and $y_2(x) = \exp(-3x)$ are linearly independent solutions of the differential equation

$$y'' + 5y' + 6y = 0$$

The first step is to verify that they are solutions of the equation:

```
restart;
ode:=diff(y(x),x$2)+5*diff(y(x),x)+6*y(x);
y1:=exp(-2*x);
y2:=exp(-3*x);
simplify(subs(y(x)=y1,ode));
simplify(subs(y(x)=y2,ode));
```

Next we need to evaluate the Wronskian. MAPLE defines the Wronskian as the matrix of functions and derivatives rather than the determinant of the matrix as in the book (and elsewhere). So we need to take the determinant to get the quantity of interest. To access the determinant routine (called `det`) we need the MAPLE linear algebra routines which we load from the `linalg` library. Then we compute the Wronskian:

```
with(linalg):
Wron:=simplify(det(Wronskian([y1,y2],x)));
```

Since `Wron` (the Wronskian from the book) is non zero, the two solutions are independent. If the Wronskian were zero, the solutions would be dependent and not form a fundamental set.

6.2 Solving Second Order Equations

6.2.1 Using the characteristic equation

Maple can help you to compute the roots of the characteristic equation for an ordinary differential equation. Consider the equation

$$\begin{aligned} 4y'' - 8y' + 3y &= 0 \\ y(0) = 2 & \qquad y'(0) = 1/2 \end{aligned}$$

This is Example 3 on page 119 of your book (so you can check your answer is correct). The characteristic equation for this differential equation is

$$4r^2 - 8r + 3 = 0$$

We can enter this into Maple and solve it as follows:

```
4*r^2-8*r+3=0;
solve('',r);
```

The solution should be $r = 1/2$ and $r = 3/2$. Thus the general solution of the ordinary differential equation is

$$y(x) = a\exp(x/2) + b\exp(3x/2).$$

We can enter this into Maple and find a and b using the initial conditions. Notice that we use `diff` to compute y' and `subs` to substitute $x = 0$ into y and y'.

```
y:=a*exp(x/2)+b*exp(3*x/2);
ic1:=subs(x=0,y)=2;
ic2:=subs(x=0,diff(y,x))=1/2;
solve({ic1,ic2},{a,b});
```

Now we can assign the computed values to a and b, look at the solution and plot the solution

```
assign('');
y;
plot(y,x=0..3);
```

6.2.2 Using `dsolve`

An easier way is to get the solution is to use `dsolve` to compute it directly. Suppose we want to compute the solution of the same second order initial value problem as in the previous section:

$$\begin{aligned} 4y'' - 8y' + 3y &= 0 \\ y(0) = 2 & \qquad y'(0) = 1/2 \end{aligned}$$

We start by entering the differential equation (the `restart` is there to make sure that previous calculations don't pollute the current one):

```
restart;
ode:=4*diff(diff(y(x),x),x)-8*diff(y(x),x)+3*y(x)=0;
ic:=y(0)=2,D(y)(0)=1/2;
```

Note that in entering the differential equation we have use `diff(y(x),x)` to represent y' and `diff(diff(y(x),x),x)` to represent y''. However, in setting the initial conditions we write $y'(0)$ as $D(y)(0)$. This inconsistency is just one of Maples delightful quirks. After entering the equation and initial conditions we can compute the solution:

```
soln:=dsolve({ode,ic},y(x));
```

The result should be

$$soln := \mathrm{y}(\,x\,) = \frac{5}{2}\,\mathrm{e}^{(\,1/2\,x\,)} - \frac{1}{2}\,\mathrm{e}^{(\,3/2\,x\,)}$$

As you can see `soln` is set to the solution $y(x) = \cdots$. In order to plot the solution we must extract the right hand side of the equality (i.e. the function). This can by done using the `rhs` command on the expression `soln`). Then we can plot the solution:

```
y:=rhs(soln);
plot(y,x=0..3);
```

You can now print a copy of the graph by using the *Print* command in the *File* menu. It's best to choose *PostScript* as the plotting language. This will create a file called `plotoutfile.ps` in your file space. You can then print the file by typing the standard UNIX printing command on your system into an a window (not Maple). In the mathematical sciences lab. the command is `lpr plotoutfile.ps` whereas on the composers it is `qpr -q smips plotoutfile.ps` (to print the file in Smith).

Note that there is another way of entering the ODE rather than using `diff(diff(`. You can use

```
restart;
ode:=4*diff(y(x),x$2)-8*diff(y(x),x)+3*y(x)=0;
```

The `x$2` tells MAPLE that you are using the second derivative.

6.3 Complex roots of the Characteristic equation

The MAPLE routine `dsolve` can solve second order problems in which the roots of the characteristic equation are complex. Suppose we want to compute the solution of the following second order initial value problem (Example 3 on page 140 of your book):

$$\begin{aligned} 16y'' - 8y' + 145y &= 0 \\ y(0) = -2 \qquad & y'(0) = 1 \end{aligned}$$

Let us first find the general solution. We start by entering the differential equation:

```
ode:=16*diff(diff(y(x),x),x)-8*diff(y(x),x)+145*y(x)=0;
```

Now `dsolve` will give you the solution:

```
soln:=dsolve(ode,y(x));
```

The result should be

$$soln := \mathrm{y}(\,x\,) = \_C1\, \mathrm{e}^{(1/4\,x)} \cos(\,3\,x\,) + \_C2\, \mathrm{e}^{(1/4\,x)} \sin(\,3\,x\,)$$

As you can see `soln` is set to the solution written using only *real* functions and coefficients.

Now you can find the constants `_C1` and `_C2`. We substitute the initial conditions into the right hand side and solve for the contants. Then we assign values to the constants:

```
eq1:=subs(x=0,rhs(soln))=-2;
eq2:=subs(x=0,diff(rhs(soln),x))=1;
solve({eq1,eq2},{_C1,_C2});
assign(");
```

Finally lets plot the solution:

```
plot(rhs(soln),x=0..8);
```

The plot should look like Figure 3.4.2 on page 141. As you can see the solution oscillates back and forth and also the amplitude (or height) of the oscillations grows.

A quicker way to do the problem would have been for `dsolve` to find the solution directly (rather than via the general solution). For example:

```
restart;
ode:=16*diff(y(x),x$2)-8*diff(y(x),x)+145*y(x)=0;
ic:=y(0)=-2,D(y)(0)=1;
soln:=dsolve({ode,ic},y(x));
plot(rhs(soln),x=0..8);
```

Now have MAPLE solve the following problem and plot the solution (the difference with the previous case is that this sign of the coefficient of y' has changed):

$$\begin{aligned} 16y'' + 8y' + 145y &= 0 \\ y(0) = -2 \qquad & y'(0) = 1 \end{aligned}$$

This solution should be oscillatory, but with decreasing amplitude.

Finally solve and plot the solution of the following differential equation (with no term involving y'):

$$\begin{aligned} 16y'' + 145y &= 0 \\ y(0) = -2 \qquad & y'(0) = 1 \end{aligned}$$

This solution should oscillate with a constant amplitude.

Chapter 7

Second Assignment (for credit)

7.1 Second order linear ordinary differential equations

For each of the following questions hand in a MAPLE session file (printed using *File/Print*) which shows the commands you used to obtain the requested output. Clearly mark (in pen if you like) the solution for each problem. Obtain a graph of $y(x)$ against x for the indicated interval of time and indicated equations. Hand in a print of the solution graph (mark clearly which graph belongs to which solution!).

The following examples show some of the behavior found in linear second order ordinary differential equations (with constant coefficients). We will analyze these problems during the next few weeks and understand each result.

1.
$$\begin{aligned} y'' + 5y' + 6y &= 0 \\ y(0) = 2 \qquad & y'(0) = 3 \end{aligned}$$
For $0 \le x \le 3$.

2.
$$\begin{aligned} y'' + 2y' + 5y &= 0 \\ y(0) = 1 \qquad & y'(0) = 0 \end{aligned}$$
For $0 \le x \le 4$.

3.
$$\begin{aligned} 9y'' - 12y' + 4y &= 0 \\ y(0) = 2 \qquad & y'(0) = -1 \end{aligned}$$
For $0 \le x \le 10$.

4.

$$\begin{aligned} y'' + y &= 0.5\cos(0.8x) \\ y(0) = 0 \qquad & y'(0) = 0 \end{aligned}$$

For $0 \leq x \leq 100$.

5.

$$\begin{aligned} y'' + y &= 0.5\cos(x) \\ y(0) = 0 \qquad & y'(0) = 0 \end{aligned}$$

For $0 \leq x \leq 40$. How do you know that the solution obtained by MAPLE is incorrect? Use the `numeric` option to `dsolve` to compute a numerical solution to the problem and compare the graph with the standard `dsolve` solution.

6.

$$\begin{aligned} y'' + 0.125y' + y &= 3\cos(2x) \\ y(0) = 0 \qquad & y'(0) = 0 \end{aligned}$$

For $0 \leq x \leq 50$.

Chapter 8

Inhomogeneous Linear Problems

For this handout I show Maple's response to some of the commands since it's rather difficult to follow without hardcopy.

8.1 The Method of Undetermined Coefficients

Maple can help you to do the *Method of Undertermined Coefficients*. First you need to have Maple calculate the fundamental set so you can decide what particular solution to guess. For example suppose that the ODE is given by

$$y'' + y = \cos(x)/2$$

We enter it as usual into Maple:

```
ode:=diff(y(x),x$2)+y(x)=cos(x)/2;
```

which produces the following output in Maple:

$$ode := \left(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(\,x\,)\right) + \mathrm{y}(\,x\,) = \frac{1}{2}\cos(\,x\,)$$

We want to find the solution of the homogeneous problem, so we need to the right hand side to be zero (hence we select the left hand side using `lhs`) and set it equal to zero):

```
Yc:=dsolve(lhs(ode)=0,y(x));
```

Maple responds with:

$$Yc := \mathrm{y}(\,x\,) = \_C1\,\cos(\,x\,) + \_C2\,\sin(\,x\,)$$

Note that the right hand side of the inhomogeneous differential equation is a solution of the homogeneous problem so we must make the following guess for the particular solution:

```
Yp:=x*(A*cos(x)+B*sin(x));
```

Maple responds with:

$$Yp := x\,(\,A\cos(\,x\,) + B\sin(\,x\,)\,)$$

Now we substitute this guess into the differential equation

```
subs(y(x)=Yp,ode);
```

to obtain

$$\left(\frac{\partial^2}{\partial x^2}\,x\,(\,A\cos(\,x\,) + B\sin(\,x\,)\,)\right) + x\,(\,A\cos(\,x\,) + B\sin(\,x\,)\,) = \frac{1}{2}\cos(\,x\,)$$

Now we can simplify the result to force the derivatives to be evaluated

```
eqn:=simplify(");
```

and obtain

$$eqn := -2\,A\sin(\,x\,) + 2\,B\cos(\,x\,) = \frac{1}{2}\cos(\,x\,)$$

It turns to be necessary to collect all the terms on the left hand side and then solve for A and B

```
eq:=lhs(eqn)-rhs(eqn)=0;
solve({coeff(lhs(eq),cos(x))=0,coeff(lhs(eq),sin(x))=0},{A,B});
```

Maple gives the following solution

$$\left\{B = \frac{1}{4}, A = 0\right\}$$

Now we set A and B to the values calculated previously

```
assign(");
```

So now we have the desired general solution

```
gsoln:=rhs(Yc)+Yp;
```

which MAPLE reports as follows

$$gsoln := \_C1\cos(\,x\,) + \_C2\sin(\,x\,) + \frac{1}{4}\,x\sin(\,x\,)$$

Suppose we also want to satisfy $y(0) = 0$ and $y'(0) = 0$. We need to have Maple compute _C1 and _C2. First we enforce $y(0) = 0$ and then $y'(0) = 0$

```
eq1:=simplify(subs(x=0,gsoln))=0;
eq2:=simplify(subs(x=0,diff(gsoln,x)))=0;
solve({eq1,eq2},{_C1,_C2});
```

which gives the uninspiring solution

$$\{\,\_C1 = 0, \_C2 = 0\,\}$$

Finally we can substitute these values into the general solution

```
soln:=subs(",gsoln);
```

to get the solution

$$soln := \frac{1}{4}\, x \sin(\, x\,)$$

MAPLE can also get the solution directly using `dsolve`. Here is how

```
ic:=y(0)=0,D(y)(0)=0;
soln1:=dsolve({ode,ic},y(x));
```

On our Sun system under Maple V R3 we get

$$soln1 := \mathrm{y}(\,x\,) = \frac{1}{2}\sin(\,x\,)\arctan\left(\frac{\sin(\,x\,)}{\cos(\,x\,)+1}\right)$$

Lets compare the two solutions by plotting them coloring our correct one in red and MAPLE's `dsolve` solution in blue

```
p1:=plot(soln,x=0..10,color=red):
p2:=plot(rhs(soln1),x=0..10,color=blue):
with(plots):
display({p1,p2});
```

Note that MAPLEs solution agrees with our solution until $x = \pi$. MAPLE's soluton is incorrect after that because it makes a mistake about evaluating arctan which is a multi-valued function.

8.2 Variation of Parameters

MAPLE can also do *variation of parameters.* Again we will solve the differential equation

$$y'' + y = \frac{1}{2}\cos(x)$$

First we need to calculate the general solution (as in the previous section). If you have already set `Yc` to the general solution (in the previous section) skip this step. Otherwise type:

```
ode:=diff(y(x),x$2)+y(x)=cos(x)/2;
Yc:=dsolve(lhs(ode)=0,y(x));
```

The first step in variation of parameters is to get the fundamental set from the general solution `Yc` computed above. This is done as follows:

```
y1:=rhs(subs(_C1=1,_C2=0,Yc));
```

so

$$y1 := \cos(x)$$

and

```
y2:=rhs(subs(_C1=0,_C2=1,Yc));
```

so

$$y2 := \sin(x)$$

Next we set the function g equal to the inhomogeneous term:

```
g:=rhs(ode);
```

and Maple shows

$$g := \frac{1}{2}\cos(x)$$

We also need the Wronskian, which MAPLE will compute for you if you first load the `linalg` package

```
with(linalg):
Wron:=simplify(det(Wronskian([y1,y2],x)));
```

and Maple correctly reports

$$Wron := 1$$

Finally we can use the Variation of Parameters formula

```
Yp:=simplify(-y1*int(y2*g/Wron,x)+y2*int(y1*g/Wron,x));
```

and obtain a particular solution:

$$Yp := \frac{1}{4}\,x\sin(x).$$

This is a perfectly good particular solution and could be used to write down the general solution etc.

8.3 Homogeneous Second Order Problems

In this exercise, we shall investigate the effects of changing the damping parameter b in the ordinary differential equation

$$ay'' + by' + cy = 0$$

(The damping parameter is the coefficient in front of the first derivative term). First enter the general second order homogeneous linear differential equation (given above) into MAPLE. Then we will look at the solution for some special choices of the coefficients.

```
ode:=a*diff(y(x),x$2)+b*diff(y(x),x)+c*y(x)=0;
```

8.3.1 No damping

The first example has $a = 1$, $b = 0$ and $c = 1$. Since $b = 0$ there is no damping and with this choice of parameters $b^2 - 4ac < 0$ so we expect an pure oscillatory solutiom. To compute this, we start by substituing the values for a, b and c:

```
ode1:=subs(a=1,b=0,c=1,ode);
```

Next we will choose some arbitrary non-zero initial conditions, $y(0) = 1$ and $y'(0) = 0$:

```
ic:=y(0)=1,D(y)(0)=0;
```

Next we can use `dsolve` to find the solution.

```
soln1:=dsolve({ode1,ic},y(x));
```

There is not much need to plot this solution, but we can do so anyway. Remember to use the `rhs` function to pick the right hand side as the thing to plot.

```
plot(rhs(soln1),x=0..10);
```

Here is the solution for another set of initial conditions

```
ic:=y(0)=0,D(y)(0)=1;
soln1a:=dsolve({ode1,ic},y(x));
```

Again we can plot the solution. This ordinary differential equation has solutions that oscillate for ever. There is no damping.

```
plot(rhs(soln1a),x=0..10);
```

8.3.2 Negative damping

In this case we choose $a = 16$, $b = -8$ and $c = 145$. Since $b < 0$, this example has solutions that grow in amplitude due to the fact that the roots of the characteristic equation have positive real part. We can input the differential equation as follows:

```
ode2:=subs({a=16,b=-8,c=145},ode);
```

All we need to demonstrate the growing solution is to have some arbitrary non-zero initial data so we choose $y(0) = 0$ and $y'(0) = 1$:

```
ic:=y(0)=0,D(y)(0)=1;
```

The solution has grows exponentially as you can tell from the next plot:

```
soln2:=dsolve({ode2,ic},y(x));
plot(rhs(soln2),x=0..10);
```

As you can see the solution grows from zero...

8.3.3 Positive Damping

The next example has solutions that die away as x increases due to the roots of the characteristic equation having negative real part. The solution is said to be damped.

Under damping:

We start by choosing $b = 1$:

```
ode3:=subs({a=1,b=1,c=1},ode);
```

Again the initial conditions are chosen arbitrarily. Here we take $y(0) = 1$, $y'(0) = 1$:

```
ic:=y(0)=1,D(y)(0)=1;
```

As we can see from the output of the next command, the solution is multiplied by a decaying exponential and so it tends to zero for large x:

```
soln3:=dsolve({ode3,ic},y(x));
plot(rhs(soln3),x=0..10);
```

The magnitude of b effects the decay of the solution. For this value of b ($b = 2$) there are no sines or cosines in the solution (no long term oscillation). If we increas b we might

Critical damping:

Increasing b causes a more rapid decrease of the solution to zero unless b is increased too far. For the value of b used in this example ($b = 2$) there are no sines or cosines in the solution (so no long term oscillation).

```
ode3:=subs({a=1,b=2,c=1},ode);
soln3:=dsolve({ode3,ic},y(x));
plot(rhs(soln3),x=0..10);
```

The solution decays more rapidly to zero than when $b = 1$.

Over damping:

If we make b larger the decay of the solution to zero slows down. Here $b = 3$.

```
ode4:=subs({a=1,b=3,c=1},ode);
soln4:=dsolve({ode4,ic},y(x));
plot(rhs(soln4),x=0..10);
```

8.3.4 Phase Angles

Next I want to illustrate phase angles . Let us consider the following undamped homogeneous differential equation with the given initial conditions

$$u'' + 4u = 0$$

with

$$u(0) = 1 \text{ and } u'(0) = 2$$

which is entered on the usual way:

```
ode:=diff(u(x),x$2)+4*u=0;
ic:=u(0)=1,D(u)(0)=2;
```

As usual we can get a solution using `dsolve` (a correct one!)

```
soln:=dsolve({ode,ic},u(x));
```

which Maple shows to be

$$soln := \mathrm{u}(\,x\,) = \cos(\,2\,x\,) + \sin(\,2\,x\,)$$

Now let us plot the solution and compare it to a pure cosine with the same amplitude. You can see that the solution is just a translation of the simple cosine

```
p1:=plot(rhs(soln),x=0..10,color=red):
p2:=plot(sqrt(2)*cos(2*x),x=0..10,color=blue):
with(plots):
display({p1,p2});
```

The predicted (from class) phase change is $-\pi/4$. If we plot this solution it is agrees exactly with the true solution above

```
p3:=plot(sqrt(2)*cos(2*x-Pi/4),x=0..10,color=green):
display({p1,p2,p3});
```

8.4 Assignment on the solution of inhomogeneous second order problems

Hand in a MAPLE worksheet and appropriate graphs for each question.

8.4.1 Inhomogeneous problems

Consider the following initial value problem for a second order ordinary differential equation:

$$\begin{aligned} y'' + 25y &= \sin(11x/2) \\ y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

Use MAPLE to find the solution of this problem in the following three ways.

1. Use the method of *undetermined coefficients* in MAPLE.
2. Use the method of *variation of parameters* in MAPLE.
3. Use the MAPLE routine `dsolve` directly.

Plot graphs of each of these solutions for $0 \le x \le 30$. This is an example of *beating*.

8.4.2 Resonance

By graphing solutions for different values of ω, find a value of ω that maximizes the amplitude of the long terms solution (after initial transients have decayed) for

$$\begin{aligned} y'' + y' + 25y &= 10\sin(\omega x) \\ y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

Use the general solution for the problem (obtained in MAPLE) to show that your predicted value is (close to) correct.

Chapter 9

Series Solutions

9.1 Series solutions in `dsolve`

MAPLE can give you the series solution to an ordinary differential equation by using a special option in the `dsolve`. Suppose you want to find the series solution to

$$y'' - 2xy' - 2y = 0.$$

As usual you enter your differential equation and initial conditions as follows:

```
ode:=diff(y(x),x$2)-2*x*diff(y(x),x)-2*y(x)=0;
```

Here we shall use general initial conditions ($y(0) = a$ and $y'(0) = b$) so you can see the two linearly independent terms in the solution:

```
ic:=y(0)=a,D(y)(0)=b;
```

Now you can use `dsolve` with the "type=series" flag set. This will cause MAPLE to find a series solution using the default number of terms (up to order 6):

```
soln:=rhs(dsolve({ode,ic},y(x),type=series));
```

This solution is a little difficult to understand. We can regroup the terms to clarify that there are two independent solutions:

```
z:=a*coeff(convert(soln,polynom),a)+b*coeff(convert(soln,polynom),b);
```

Maple allows you to change the order of the series solution by specifying a different value of the `Order` parameter: For example to compute more terms (`Order=10` rather than the default of 6) you do the following:

```
Order:=10;
soln10:=rhs(dsolve({ode,ic},y(x),type=series));
z10:=a*coeff(convert(soln,polynom),a)+b*coeff(convert(soln,polynom),b);
```

9.2 Taking more and more terms

This short demonstration shows how taking succesively more and more terms in a series solution improves thae accuracy of the solution. First we enter a the simple differential equation

$$y'' + 25y = 0$$

and the initial conditions (chosen arbitrarily)

$$y(0) = 0 \text{ and } y'(0) = 1$$

and compute it's exact solution (which we could have done more easily be hand!):

```
ode:=diff(y(x),x$2)+25*y(x)=0;
ic:=y(0)=0,D(y)(0)=1;
exact:=rhs(dsolve({ode,ic},y(x)));
```

Maple displays the exact solution:

$$exact := \frac{1}{5}\sin(\,5\,x\,)$$

Now we plot the exact solution in white saving it for later use:

```
pe:=plot(exact,x=0..1,color=white):
```

Next we find the series solution using successively higher orders. Note we `convert` the series solution to a polynomial because we are going to plot it:

First `Order=2`:

```
Order:=2;
s2:=convert(rhs(dsolve({ode,ic},y(x),type=series)),polynom);
```

which Maple shows to e

$$s2 := x$$

now we plot it and save the picture

```
p2:=plot(s2,x=0..1,color=red):
```

Next `Order=4`:

```
Order:=4;
s4:=convert(rhs(dsolve({ode,ic},y(x),type=series)),polynom);
```

giving the solution

$$s4 := x - \frac{25}{6}\,x^3$$

and we plot this one

```
p4:=plot(s4,x=0..1,color=orange):
```

Next `Order=6`:

```
Order:=6;
s6:=convert(rhs(dsolve({ode,ic},y(x),type=series)),polynom);
```

Giving the series

$$s6 := x - \frac{25}{6}\,x^3 + \frac{125}{24}\,x^5$$

and plotting:

```
p6:=plot(s6,x=0..1,color=yellow):
```

Finally `Order=8`:

```
Order:=8; s8:=convert(rhs(dsolve({ode,ic},y(x),type=series)),polynom);
```

Which gives even more terms:

$$s8 := x - \frac{25}{6}\,x^3 + \frac{125}{24}\,x^5 - \frac{3125}{1008}\,x^7$$

and we plot this one

```
p8:=plot(s8,x=0..1,color=green):
```

Now we can superimpose the plots of the various approximate solutions. Even `Order=8` does not give an acceptable solution as far as $x = 1$!:

```
with(plots):
display({pe,p2,p4,p6,p8});
```

Chapter 10

The Laplace Transform

10.1 Computing the Laplace transform

Note that Maple can compute the Laplace transform of simple functions. For example to compute the Laplace transform of $\cos(3t)$ you type in Maple:

```
laplace(cos(3*t),t,s);
```

Here t is the variable in $\cos(3*t)$ and s is the transform variable. **From now on** you will be allowed to use Maple for Laplace and inverse Laplace transforms unless explicitly told otherwise in a question.

10.2 Computing the inverse Laplace transform

Maple can help you with the difficult problem of inverting a Laplace transform (ie computing $\mathcal{L}^{-1}$). As an example, let us consider problem 21 on page 289 in which you are two solve the initial value problem:

$$\begin{aligned} y'' - 2y' + 2y &= \cos(t) \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

If $Y(s) = \mathcal{L}(y(t))$, you can compute (by hand!) that Y is given by

$$Y(s) = \frac{\frac{s}{s^2+1} + s - 2}{s^2 - 2s + 2}$$

To find $y(t) = \mathcal{L}^{-1}(Y(s))$ type the following commands in Maple:

```
readlib(laplace):
Y:=(s/(s^{}2+1)+s-2)/(s^{}2-2*s+2);
invlaplace(Y,s,t);
```

The command `readlib(laplace)` tells Maple to load the Laplace transform library. Then Y is set to be the given function and finally `invlaplace` computes the inverse Laplace transform. The first argument s of `invlaplace` is the argument of the transform $Y(s)$ and the second argument is the argument for y.

10.3 Solving an ODE in MAPLE via Laplace transforms

You can use Maple to solve the entire problem. For simplicity we shall rewrite the differential equation as

$$y'' - 2y' + 2y - \cos(t) = 0$$

We start by entering this differential equation (with right hand side subtracted) into Maple as follows:

```
ode:=diff(y(t),t$2)-2*diff(y(t),t)+2*y(t)-cos(t);
```

Next we take the Laplace transform of the equation (and right hand side):

```
lode:=laplace(ode,t,s);
```

Note that Maple uses $y(0)$ and $D(y)(0)$ to denote the values of $y(0)$ and $y'(0)$. We substitute the correct values next:

```
lode:=subs(y(0)=1,D(y)(0)=0,lode);
```

Next we solve to obtain the Laplace transform of the solution:

```
Y:=solve(lode=0,laplace(y(t),t,s));
```

Finally we can invert the Laplace transform to obtain the solution:

```
y:=invlaplace(Y,s,t);
```

Exercise (not for credit): Use Maple to help you solve the initial value problem:

$$\begin{aligned} y'' + 2y' + y &= 4\exp(-t) \\ y(0) &= 2 \\ y'(0) &= -1 \end{aligned}$$

This is question 23 on page 289 of your book. Did you get the correct answer?

10.4 The Dirac and Heaviside functions

Maple even knows about the two new functions introduced in Chapter 6 of your book. The unit step function $u_c(t)$ is given by `Heaviside(t-c)`. Thus you should be able to guess what the following commands will do:

```
plot(Heaviside(t-2),t=0..4);
laplace(Heaviside(t-2),t,s);
```

Maple also knows about the Dirac δ function. See what happens when you type

```
readlib (laplace);
invlaplace(1,s,t);
laplace(Dirac(t-1),t,s);
```

10.5 `dsolve` and Laplace transforms

You can force `dsolve` to use the Laplace transform method. This is useful in two cases, firstly if the right hand side involves the special functions `Dirac` or `Heaviside` and secondly if their is resonance (remember `dsolve` fails if the right hand side of the differential equation is a solution of the homogeneous problem. First we demonstrate using `dsolve` with the Laplace transform on an ordinary problem:

$$\begin{aligned} y'' - 2y' + 2y &= \cos(t) \\ y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

which can be solved as follows (`soln` and `soln1` should be the same).

```
ode:=diff(y(t),t$2)-2*diff(y(t),t)+2*y(t)-cos(t);
ic:=y(0)=0,D(y)(0)=0;
soln:=rhs(dsolve({ode,ic},y(t),method=laplace));
soln1:=simplify(rhs(dsolve({ode,ic},y(t))));
```

Next we show solving a resonant problem. The Laplace transform method succeeds as the graphics show:

$$\begin{aligned} y'' + y &= \cos(t) \\ y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

which we solve as follows:

```
ode:=diff(y(t),t$2)+y(t)-cos(t);
ic:=y(0)=0,D(y)(0)=0;
soln:=rhs(dsolve(ode,y(t),method=laplace));
soln:=subs(ic,soln);
soln1:=rhs(dsolve({ode,ic},y(t)));
with(plots):
p1:=plot(soln,t=0..10,color=red):
p2:=plot(soln1,t=0..10,color=blue):
display({p1,p2});
```

Clearly the red curve is correct and the blue curve can not be a solution of the differential equation (why?). Note that it seems to be better to impose the initial conditions outside `dsolve` when using the Laplace method.

Exercise: Use Maple to help you solve the initial value problem:

$$\begin{aligned} y'' - 2y' + 2y &= \delta(t) \\ y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

The solution is called the impulse response for the ordinary differential equation. It should be

$$y(t) = \exp(t)\sin(t)$$

Curiously, `dsolve` gives the wrong answer unless you use `method=laplace`.

Chapter 11

First Order Linear Systems

11.1 Linear Algebra in MAPLE

Maple can compute eigenvalues and eigenvectors to help with your analysis of linear systems (as long as you have at most four differential equations!!). To use MAPLE's linear algebra package you should use the following two commands. The first loads the `linalg` package and the second sets `Id` to be the identity matrix (appropriate to the context).

```
with(linalg):
alias(Id=&*());
```

Matrices are entered row by row. The following command enters a 2×2 matrix

$$A = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$$

into MAPLE:

```
A:=matrix(2,2,[[2,1],[2,2]]);
```

The eigenvalues can then be computed using the `eigenvals` command, and the two eigenvalues are saved in the variables `l1` and `l2`.

```
evs:=eigenvals(A);
l1:=evs[1];
l2:=evs[2];
```

There is an `eigenvects` command to find the eigenvalues and eigenvectors of a matrix together. But I have had problems with this command (it gives strange output). So I prefer to find the eigenvectors by finding all solutions of $(A - \lambda I)\boldsymbol{x} = 0$. This is done as follows. Here `sp1` is the set of all solutions - in this case just one vector! We pick out this one vector and call it `v1`. The same thing is done to get `v2`.

```
sp1:=nullspace(l1*Id-A);
v1:=sp1[1];
sp2:=nullspace(l2*Id-A);
v2:=sp2[1];
```

We can check that these eigenvalues and eigenvectors work:

```
map(simplify,evalm(multiply(A,v1)-l1*v1));
map(simplify,evalm(multiply(A,v2)-l2*v2));
```

We can try to diagonalize A by forming a matrix from the eigenvectors (notice the transpose since the eigenvectors becomes rows not columns). Then $T^{-1}AT$ produces the diagonal matrix D as we expect:

```
T:=transpose(matrix(2,2,[evalm(v1),evalm(v2)]));
multiply(inverse(T),multiply(A,T));
```

11.2 Phase Portraits of Linear Systems of Differential Equations

The package DEtools contains two routines to help you draw phase portraits. They are `DEplot2` and `phaseportrait`. To draw the phase portrait of $\boldsymbol{x}' = A\boldsymbol{x}$, we enter the matrix into MAPLE. Here

$$A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$$

(this is the matrix from class). We can the compute the eigenvalues to find the stability and type of the equilibrium point:

```
with(linalg):with(DEtools):
A:=matrix(2,2,[[5,-1],[3,1]]);
eigenvals(A);
```

From the eigenvalues, you can try to make predicitions about the stability and type of the equilibrium point $(0,0)^T$. Is the equilibrium point stable in this case? What is the type of the equilibrium point?

To draw a phase portrait of the 2×2 linear system, `phaseportrait` allows you to simply pass the matrix to the function. Then you need to tell `phaseportrait` the dependent variables (here $[x, y]$), the integration time (this is chosen be trial and error) here we choose $t = -2..2$. Then there are a list of initial conditions of the form $\{[t_0, x_0, y_0], [t_1, x_1, y_1], \cdots\}$ and finally graphing limits to make sure we can see the graph:

```
phaseportrait(A,[x,y],t=-2..2,{[0,0.5,0],[0,1.5,0],[0,1,1],[0,-1,-1],
[0,1,3],[0,-1,-3],[0,-1,0],[0,0,1]},x=-2..2,y=-2..2);
```

One of the graphs looks a little rough, so we can force a smoother plot by demanding a smaller step size:

```
phaseportrait(A,[x,y],t=-2..2,{[0,0.5,0],[0,1.5,0],[0,1,1],[0,-1,-1],
[0,1,3],[0,-1,-3],[0,-1,0],[0,0,1]},x=-2..2,y=-2..2,stepsize=0.1);
```

The command `DEplot2` has a similar syntax, but also plots the direction field:

```
DEplot2(A,[x,y],t=-2..2,{[0,0.5,0],[0,1.5,0],[0,1,1],[0,-1,-1],[0,1,3],
[0,-1,-3],[0,-1,0],[0,0,1]},x=-2..2,y=-2..2);
```

11.2.1 Exercise - not for credit

Consider the following ordinary differential equation $\boldsymbol{x}' = A\boldsymbol{x}$ with

$$A = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}$$

- First find the eigenvalues of A and say if the equilibrium point is stable or unstable.
- Find the eigenvectors of A and hence write down the general solution. Check your result by comparing your results to Example 2 on page 354 of your book.
- Draw a phase portrait of the solution including the two fundamental solutions found above.

11.3 Systems with Complex Characteristic Roots

Use MAPLE to compute the eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$$

Note the eigenvalues are a complex conjugate pair. Now draw the phase plane (with direction field) for the differential equation $\boldsymbol{x}' = A\boldsymbol{x}$. As you can see the solutions are closed orbits around the equilibrium point.

Here are the commands you could have used to answer this question:

```
with(linalg):
A:=matrix(2,2,[[2,-5],[1,-2]]);
eigenvals(A);
with(DEtools):
DEplot2(A,[x,y],t=-5..5,{[0,0.5,0],[0,1.5,0],[0,0.1,0]},x=-2..2,y=-2..2);
```

Now repeat this process with

$$A = \begin{pmatrix} 1 & -5 \\ 1 & -2 \end{pmatrix}$$

This time the equilibrium point is a stable spiral point.

Here are the MAPLE commands you need to use:

```
A:=matrix(2,2,[[1,-5],[1,-2]]);
eigenvals(A);
with(DEtools):
DEplot2(A,[x,y],t=-5..5,{[0,0.5,0],[0,1.5,0],[0,0.1,0]},x=-2..2,y=-2..2);
```

11.4 Systems with Repeated Characteristic Roots

Use MAPLE to compute the eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & -3/4 \\ 1/3 & 1 \end{pmatrix}$$

Note the eigenvalues are a repeated pair. Now draw the phase plane (with direction field) for the differential equation $\boldsymbol{x}' = A\boldsymbol{x}$.

Here are the commands you could have used to answer this question:

```
with(linalg):
A:=matrix(2,2,[[2,-3/4],[1/3,1]]);
eigenvals(A);
eigenvects(A);
with(DEtools):
DEplot2(A,[x,y],t=-5..5,{[0,0.5,0],[0,1.5,0],[0,0,-1],[0,0,1],[0,-1,-1],
[0,3/2,1],[0,-3/2,-1]},x=-2..2,y=-2..2,stepsize=0.1);
```

11.5 Assignment on Linear Problems

Consider the linear homogeneous ordinary differential equation with constant coefficients given by

$$\boldsymbol{x}' = A\boldsymbol{x}.$$

For each of the matrices A given below do the following:

- Use MAPLE to compute the eigenvalues and eigenvectors of the matrix A. Hence say if the equilibrium point $\boldsymbol{x} = 0$ is a stable or unstable point. Say also if the point is a node, a saddle or a focus.
- Draw a phase plane diagram for each ordinary differential equation. Display at least 6 solutions that show the general trends of solutions around the origin.
- Hand in MAPLE worksheets, the phase-plane diagrams and a clear answer for each of the following systems:

1.
$$A = \begin{pmatrix} -3/2 & 1 \\ -1/4 & -1/2 \end{pmatrix}.$$

2.
$$A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}.$$

3.
$$A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}.$$

4.
$$A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}.$$

5.
$$A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}.$$

Chapter 12

First order non-linear systems

12.1 The Van der Pol Equation

In class we predicted that the equilibrium point for the Van der Pol equation:

$$\begin{aligned} x' &= y \\ y' &= a(1-x^2)y - x \end{aligned}$$

changes stability as a changes sign. Here we will investigate this graphically. Start MAPLE and enter the Van der Pol equations (notice how to enter a system as a vector of equations):

```
with(DEtools):
fcns:=[x,y];
odes:=[diff(x(t),t)-y(t)=0,diff(y(t),t)+x(t)-a*(1-x(t)^2)*y(t)];
```

We predicted that for $a \geq 0$ and $a < 2$ the equilibrium point is unstable and we shall test the case $a = 0.25$.

12.1.1 Unstable Equilibrium point

Type the following to draw a direction field and two solutions (this will take some time to run - try not to get too impatient!):

```
DEplot2(subs(a=0.25,odes),fcns,t=0..100,{[0,0.01,0],[0,0,5]},
x=-5..5,y=-5..5,stepsize=0.1);
```

Notice how the trajectory winds out from the equilibrium point as we expect for an unstable center. Far away from the center, non-linear effects take over and the trajectory approaches a closed curve (call a *limit cycle*) which is a periodic or cyclic solution of the differential equation. A solution starting outside the limit cycle winds in closer and closer to the limit cycle. Thus if you start anywhere (ie any initial condition) except $\boldsymbol{x} = 0$, you will wind up (at large time) very close to the limit cycle. The solution is then oscillatory. To see this graph the actual solution $x(t)$ by typing the following (we tell `dsolve` to find the solution numerically, in which case we must use `odeplot` rather than `plot` to plot the solution):

```
fcns:={x(t),y(t)};
odes:=diff(x(t),t)-y(t)=0,diff(y(t),t)+x(t)-a*(1-x(t)^2)*y(t);
soln:=dsolve({odes[1],subs(a=0.25,odes[2]),x(0)=0,y(0)=5},fcns,
type=numeric);
with(plots);
odeplot(soln,[t,x(t)],0..50,numpoints=200);
```

As you can see the solution settles down to a periodic motion.

12.1.2 The critical case $a = 0$

Now what about $a = 0$? In this case the origin is a center and so our theory tells us nothing. But for the Van der Pol equation it is easy to see that when $a = 0$

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned}$$

This is a linear equation with a center at the origin. See what the direction field and some solutions look like by typing:

```
fcns:=[x,y];
odes:=[diff(x(t),t)-y(t)=0,diff(y(t),t)+x(t)-a*(1-x(t)^2)*y(t)];
DEplot2(subs(a=0,odes),fcns,t=0..100,{[0,0,1],[0,0,2],[0,0,3]},
x=-5..5,y=-5..5,stepsize=0.1);
```

12.1.3 Stable equilibrium point

Finally we turn to the case $a < 0$ and we shall choose $a = -.25$. In this case we expect a stable spiral equilibrium point. Verify this by typing

```
DEplot2(subs(a=-0.25,odes),fcns,t=0..100,{[0,1,0],[0,2,0],
[0,2.1,0]},x=-5..5,y=-5..5,stepsize=0.1);
```

Note also what happens to a trajectory starting outside the (now unstable) limit cycle.

12.2 The Predator-Prey System

In this assignment we will analyze the following predator-prey system:

$$\begin{aligned} x' &= x(1-y/2) \\ y' &= y(-3+x)/4 \end{aligned}$$

We shall do the following:

1. Calculate the equilibrium points
2. Classify the equilibrium points
3. Draw the phase plane

12.2.1 Computing the equilibrium points.

We enter the differential equations as usual and then compute all the equilibrium points by solving the equations $x' = y' = 0$ using the MAPLE `solve` command.

```
with(DEtools):
fcns:=[x,y];
odes:=[diff(x(t),t)=x*(1-y/2),diff(y(t),t)=y*(-3+x)/4];
solve({rhs(odes[1])=0,rhs(odes[2])=0},{x,y});
```

The result is two equilibrium points

$$(x, y) = (0, 0) \text{ and } (x, y) = (3, 2)$$

These could have been found by hand!!

12.2.2 Computing stability of the equilibrium points.

Now for each equilibrium point, we need to compute the Jacobian (using the MAPLE command `jacobian`) at the equilibrium point, then compute the eigenvalues to determine the type and stability of the equilibrium point. This is done as follows, first for $(x, y) = (0, 0)$

```
with(linalg):
J:=jacobian([rhs(odes[1]),rhs(odes[2])],[x,y]);
J1:=subs(x=0,y=0,evalm(J));
eigenvals(J1);
```

Thus $(x, y) = (0, 0)$ is a saddle point (we could also compute the eigenvectors to find out the stable and unstable directions). For the second point $(x, y) = (3, 2)$ we proceed as follows:

```
J2:=subs(x=3,y=2,evalm(J));
eigenvals(J2);
```

Since the eigenvalues are $r_1 = \sqrt{3}i/2$ and $r_2 = -\sqrt{3}i/2$ the linearized problem possesses a center and we can not conclude anything about the non-linear problem.

12.2.3 Plotting the solution

Drawing the phase-plane and direction fields: As usual we can use `DEplot2` to draw the phase plane and direction field. In other cases you will have to experiment with the range of t (here `t=-10..10`), the initial conditions (here $\{[0, 2.5, 2], [0, 2, 2], [0, 1, 2], [0, 0.1, 2]\}$), the graphing window (here `x=0..6,y=0..6`) and the output step size (here `stepsize=0.1`) to obtain a good picture.

```
DEplot2(odes,fcns,t=-10..10,{[0,2.5,2],[0,2,2],[0,1,2],
[0,0.1,2]},x=0..6,y=0..6,stepsize=0.1);
```

Note that there are periodic solutions but they are not limit cycles.
Plotting the solution against time: You can also plot the solutions against time in the usual way (but it seems easiest to re-enter the ode!):

```
fcns:={x(t),y(t)};
odes:=diff(x(t),t)=x(t)*(1-y(t)/2),diff(y(t),t)=y(t)*(-3+x(t))/4;
soln:=dsolve({odes[1],odes[2],x(0)=0,y(0)=5},{x(t),y(t)},type=numeric);
with(plots):
odeplot(soln,[[t,x(t)],[t,y(t)]],0..30,numpoints=200);
```

12.3 Exercise on non-linear systems

The following population model governs two competing species:

$$x' = (1 - x - y)x$$
$$y' = (2 - x - 3y)y$$

- Find all equilibrium points for this system for which $x \geq 0$ and $y \geq 0$.
- For each equilibrium point use MAPLE to compute the eigenvalues and eigenvectors of the Jacobian matrix. Hence say if the equilibrium point is a stable or unstable point. Say also if the point is a node, a saddle or a focus.
- Draw a phase plane diagram for the ordinary differential equation. Display at least 8 solutions that show the general trends of solutions.
- With this choice of model parameters, can the species co-exist? Will one wipe the other out or will both die out?
- Hand in MAPLE worksheets, the phase-plane diagrams and a clear answer.

Chapter 13

Fourier Sine Series and the Wave Equation

In class we solved the wave equation and showed (if $c = 1$) that the solution is

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(n\pi t) + b_n \sin(n\pi t)) \sin(n\pi x) \tag{13.1}$$

where a_n and b_n ($n = 1, 2, \cdots$) have to be determined. If the initial data is

$$u(x,0) = g(x) \text{ and } u_t(x,0) = h(x)$$

where $h(x)$ and $g(x)$ are known functions we proceed as follows: We plug in $t = 0$ into (13.1) to get

$$\sum_{n=1}^{\infty} a_n \sin(n\pi x) = g(x)$$

and hence (see your class notes)

$$a_n = 2 \int_0^1 g(x) \sin(n\pi x)\, dx$$

13.1 The Fourier Series of a Continuous Function

Suppose that

$$g(x) = \begin{cases} x & \text{if } 0 \le x \le 1/2 \\ (1-x) & \text{if } 1/2 < x \le 1 \end{cases}$$

then

$$a_n = 2\left(\int_0^{1/2} x \sin(n\pi x)\, dx + \int_{1/2}^{1} (1-x) \sin(n\pi x)\, dx \right).$$

We evaluate this and look at the series in MAPLE. First we just evaluate the integrals:

```
am:=simplify(2*(int(x*sin(m*Pi*x),x=0..1/2)+int((1-x)*sin(m*Pi*x),x=1/2..1)));
```

Now we can define the partial sum `gs` by

```
gs:=Sum(am*sin(m*Pi*x),m=1..M);
```

Notice we have used the "inert" form of the `sum` command to delay doing the sum until later when we will supply an M and x. Let us plot some partial sums (using $M = 1, 10$ and 20) as well as the exact function $g(x)$ which we can express conveniently using the Heaviside function:

```
with(plots):
p1:=plot(subs(M=1,gs),x=0..1,color=red):
p5:=plot(subs(M=5,gs),x=0..1,color=blue):
p20:=plot(subs(M=20,gs),x=0..1,color=green):
g:= (1-Heaviside(x-1/2))*x+Heaviside(x-1/2)*(1-x);
ge:=plot(g,x=0..1,color=white):
display({p5,p1,p20,ge});
```

As you can see, when $M = 20$ the series solution is quite close to the "hat" function (displayed in white).

You might wonder what happens if you use values of x outside $[0, 1]$. Since the sine functions in the series are periodic of period 2π and odd, the hat repeats as odd function with period 2 as shown in the next computation:

```
plot(subs(M=20,gs),x=-3..3,color=green);
```

13.2 The Fourier Series of a Discontinuous Function

Now we turn to the time derivative initial data $u_t(x, 0) = h(x)$. If you differentiate (13.1) with respect to t and plug in $t = 0$ you obtain:

$$\sum_{n=1}^{\infty} n\pi b_n \sin(n\pi x) = h(x)$$

from which we see that

$$n\pi b_n = 2\int_0^1 h(x)\sin(n\pi x)\,dx$$

In this case we shall suppose that

$$h(x) = \begin{cases} 1 & \text{if } 0 \le x \le 3/4 \\ 0 & \text{if } 3/4 < x \le 1 \end{cases}$$

then

$$n\pi b_n = 2\int_0^{3/4} \sin(n\pi x)\,dx.$$

Again we can compute the coefficients b_n as follows

```
bm:=simplify(2*int(sin(m*Pi*x),x=0..3/4)/m/Pi);
```

and define an approximation to $h(x)$ by

```
hs:=Sum(m*Pi*bm*sin(m*Pi*x),m=1..M);
```

Now the convergence of the series to $h(x)$ is much slower as shown in the following plots of partial sums of the Fourier series:

```
p10:=plot(subs(M=10,hs),x=0..1,color=red):
p20:=plot(subs(M=20,hs),x=0..1,color=blue):
h:=1-Heaviside(x-3/4);
he:=plot(h,x=0..1,color=white):
display({p10,p20,he});
```

Even adding the first 50 terms doesn't get you very close to the step function. Notice also the "bumps" close to $x = 0$ and $x = .75$. These are always present (for any M but they move closer and closer to the discontinuity as M increases) and are an example of the Gibbs phenomenon:

```
s50:=evalf(subs(M=50,hs)):
p50:=plot(s50,x=0..1,color=green):
display({p10,p20,p50,he});
```

13.3 The Solution of the Wave Equation

Now that you have computed the coefficients a_n and b_n you can compute a partial sum approximation to $u(x,t)$ by

```
u:=Sum((am*cos(m*Pi*t)+bm*sin(m*Pi*t))*sin(m*Pi*x),m=1..M);
```

With this solution you can plot u at different values of t. For example to plot an approximation to u at $t = 0.25$ using $M = 20$ terms you could type:

```
u25:=evalf(subs(t=.25,M=20,u)):
plot(u25,x=0..1);
```

You could even make an animation of the solution by typing

```
ua:=evalf(subs(M=20,u)):
animate(ua,x=0..1,t=0..2,frames=50);
```

Appendix A

Solutions to Computer Exercises

A.1 Assignment 1 answers

A.1.1 Logistic Model

The following are the Maple commands for question 1 (logistic model):

```
with(DEtools):with(plots):
ode:=diff(p(t),t)=2*p(t)-p(t)^2;
```

which produces the output:

$$ode := \frac{\partial}{\partial t}\,\mathrm{p}(\,t\,) = 2\,\mathrm{p}(\,t\,) - \mathrm{p}(\,t\,)^2$$

Then

```
p1:=dfieldplot(ode,[t,p],t=0..7,p=0..4):
p2:=DEplot(ode,[t,p],t=0..7,{[0,.1],[0,4]}):
display({p1,p2});
```

This creates the graph in Figure A.1. From the graph we can see that if $p(0) > 0$ the $p(t) \rightarrow 2$ as $t \rightarrow \infty$.

A.1.2 Spruce Budworm Model

Maple commands for the second question (spruce budworm model):

```
restart;with(plots):with(DEtools):
ode:=diff(p(t),t)=p(t)*(2-.2*p(t)-5*p(t)/(2+p(t)^2));
```

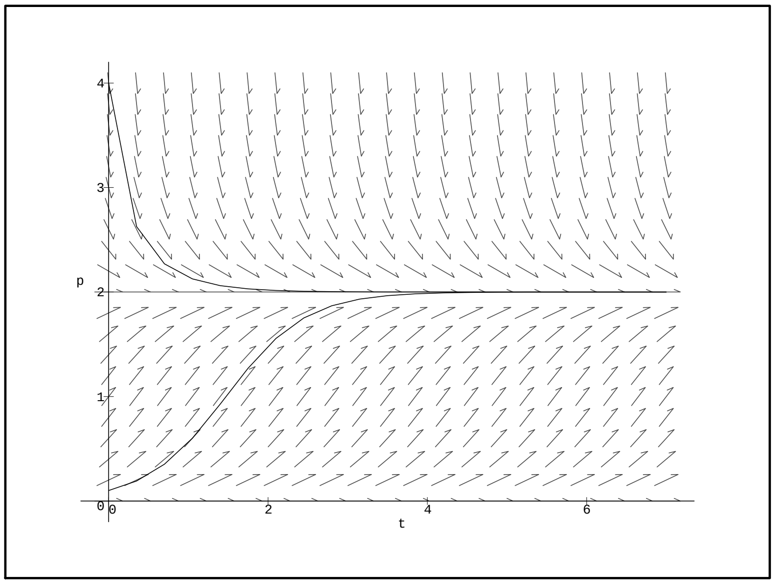


Figure A.1: Direction field and selected solutions for the logistic equation

which produces the output:

$$ode := \frac{\partial}{\partial t}\,\mathrm{p}(t) = \mathrm{p}(t)\left(2 - .2\,\mathrm{p}(t) - 5\,\frac{\mathrm{p}(t)}{2+\mathrm{p}(t)^2}\right)$$

Then

```
p1:=dfieldplot(ode,[t,p],t=0..5,p=0..8):
p2:=DEplot(ode,[t,p],t=0..5,{[0,.5],[0,2],[0,4],[0,8]}):
display({p1,p2});
```

This creates the graph in Figure A.2. To understand the solutions better we perform the following commands:

```
plot(rhs(ode),p=0..8);
solve(rhs(ode)=0,p(t));
```

This produces the graph in Figure A.3 and the following solutions of $dp/dt = 0$:

$$0, 6.124885419 - .1\,10^{-9}\,I, 1.238442818 + .1\,10^{-9}\,I,$$
$$2.636671762 + .1\,10^{-9}\,I$$

From the two graphs (Figures A.2 and A.3) and above solution, if $0 < p(0) < 2.63667\cdots$ then $p(t) \to 1.238\cdots$ as $t \to \infty$. If $p(0) > 2.63667\cdots$ then $p(t) \to 6.124885\cdots$ as $t \to \infty$. If $p(0) = 2.63667\cdots$ then $p(t) = 2.63667\cdots$ for all t (since it is an equilibrium point).

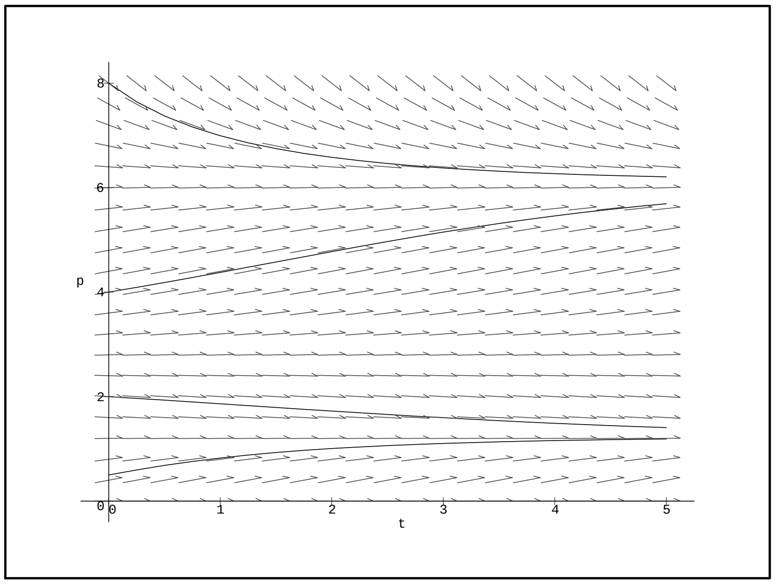


Figure A.2: Direction field and selected solutions for the spruce bud worm equation

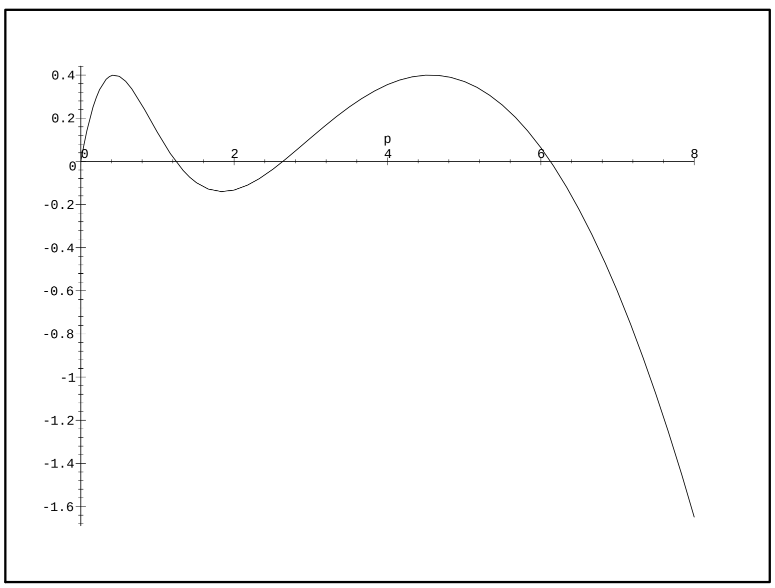


Figure A.3: A plot of dp/dt against p for the spruce budworm model.

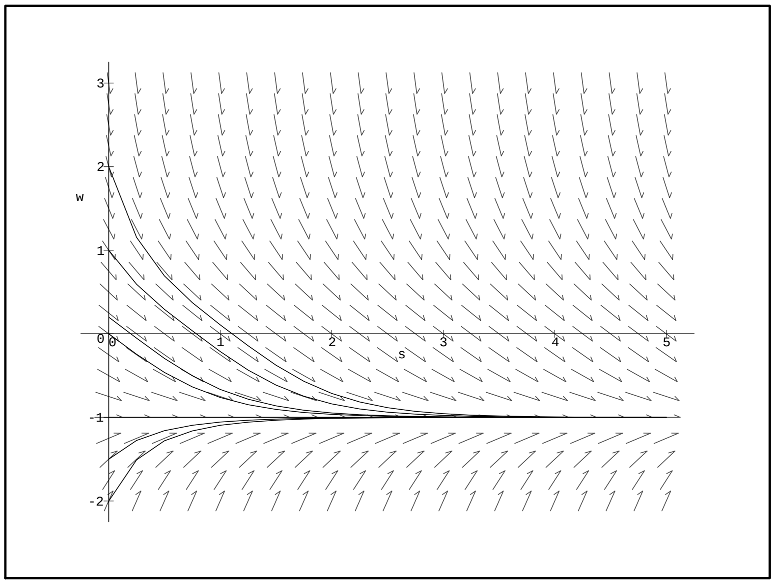


Figure A.4: Direction field and selected solutions for the falling raindrop model

A.1.3 Falling Raindrop

Maple commands for the last question (falling raindrop):

```
restart;with(plots):with(DEtools):
ode:=diff(w(s),s)=-1-w(s)*abs(w(s));
```

which produces the output:

$$ode := \frac{\partial}{\partial s}\,\mathrm{w}(\,s\,) = -1 - \mathrm{w}(\,s\,)\;|\mathrm{w}(\,s\,)|$$

Then

```
p1:=dfieldplot(ode,[s,w],s=0..5,w=-2..3):
inits:={[0,-2],[0,-1.5],[0,-1],[0,.2],[0,0],[0,1],[0,2]}:
p2:=DEplot(ode,[s,w],s=0..5,inits):
display({p1,p2});
```

This creates the graph in Figure A.4. From the graph, for any $w(0)$, we can see that $w(s) \to -1$ as $s \to \infty$. Hence $v(t) \to \sqrt{(mg/k)}$ as $t \to \infty$.

Derivation of the equation

From the fact that $t = (m/kg)^{1/2}s$ we have that $dt = (m/kg)^{1/2}ds$ so the differential equation becomes

$$\left(\frac{kg}{m}\right)^{1/2}\frac{dv}{ds} = -g - \frac{k}{m}v|v|$$

Now using $v = (mg/k)^{1/2}$ we get

$$\left(\frac{kg}{m}\right)^{1/2}\left(\frac{mg}{k}\right)^{1/2}\frac{dw}{ds} = -g - \frac{k}{m}\left(\frac{mg}{k}\right)w|w|$$

Simplifying this gives the desired differential equation.

A.2 Assignment 2 answers

A.2.1 Problem 1: Distinct real roots

Here are the MAPLE commands for this problem, together with the output and requested graph (see Figure A.5):

```
with(DEtools):with(plots):
ode1:=diff(y(x),x$2)+5*diff(y(x),x)+6*y(x)=0;
```

$$ode1 := \left(\frac{\partial^2}{\partial x^2}\mathrm{y}(x)\right) + 5\left(\frac{\partial}{\partial x}\mathrm{y}(x)\right) + 6\,\mathrm{y}(x) = 0$$

```
soln1:=dsolve({ode1,y(0)=2,D(y)(0)=3},y(x));
```

$$soln1 := \mathrm{y}(x) = -7\,\mathrm{e}^{(-3\,x)} + 9\,\mathrm{e}^{(-2\,x)}$$

```
plot(rhs(soln1),x=0..3);
```

A.2.2 Problem 2: Complex roots with damping

```
ode2:=diff(y(x),x$2)+2*diff(y(x),x)+5*y(x)=0;
```

$$ode2 := \left(\frac{\partial^2}{\partial x^2}\mathrm{y}(x)\right) + 2\left(\frac{\partial}{\partial x}\mathrm{y}(x)\right) + 5\,\mathrm{y}(x) = 0$$

```
soln2:=dsolve({ode2,y(0)=1,D(y)(0)=3},y(x));
```

$$soln2 := \mathrm{y}(x) = \mathrm{e}^{(-x)}\cos(2\,x) + 2\,\mathrm{e}^{(-x)}\sin(2\,x)$$

```
plot(rhs(soln2),x=0..4);
```

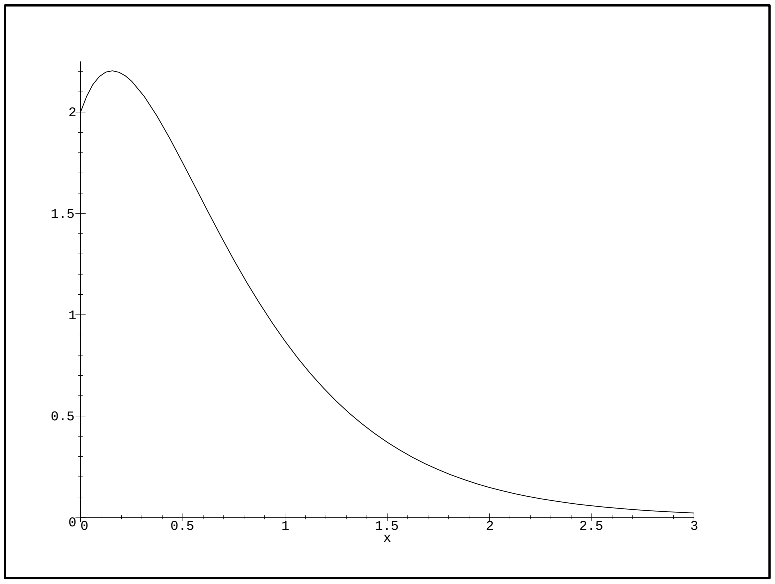


Figure A.5: The graph for assignment 2, problem 1

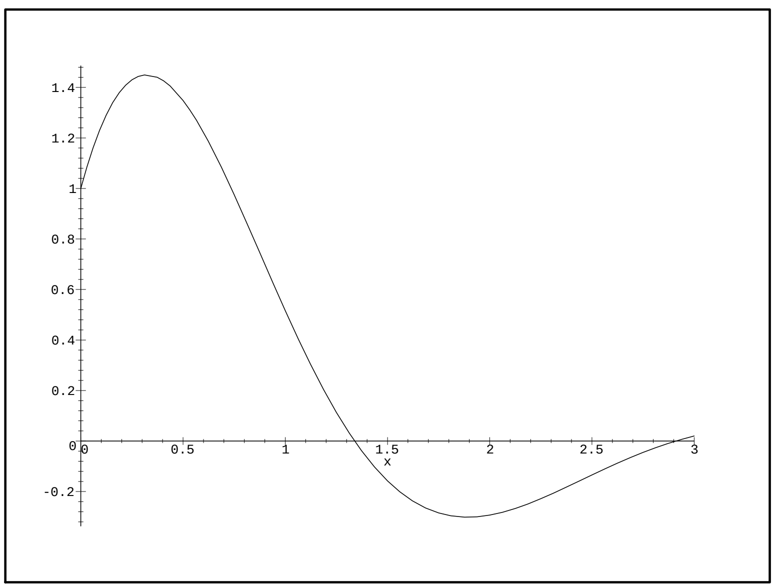


Figure A.6: The graph for assignment 2, problem 2

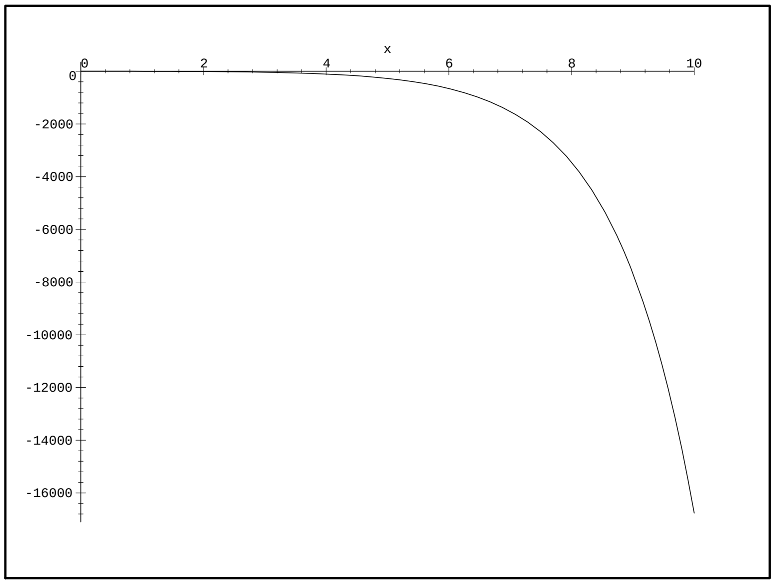


Figure A.7: The graph for assignment 2, problem 3

A.2.3 Problem 3: Distinct real roots

```
ode3:=9*diff(y(x),x$2)-12*diff(y(x),x)+4*y(x)=0;
```

$$ode3 := 9\left(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(\,x\,)\right) - 12\left(\frac{\partial}{\partial x}\,\mathrm{y}(\,x\,)\right) + 4\,\mathrm{y}(\,x\,) = 0$$

```
soln3:=dsolve({ode3,y(0)=2,D(y)(0)=-1},y(x));
```

$$soln3 := \mathrm{y}(\,x\,) = 2\,\mathrm{e}^{(\,2/3\,x\,)} - \frac{7}{3}\,\mathrm{e}^{(\,2/3\,x\,)}\,x$$

```
plot(rhs(soln3),x=0..10);
```

A.2.4 Problem 4: Beating

```
ode4:=diff(y(x),x$2)+y(x)=cos(4*x/5)/2;
```

$$ode4 := \left(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(\,x\,)\right) + \mathrm{y}(\,x\,) = \frac{1}{2}\cos\left(\frac{4}{5}\,x\right)$$

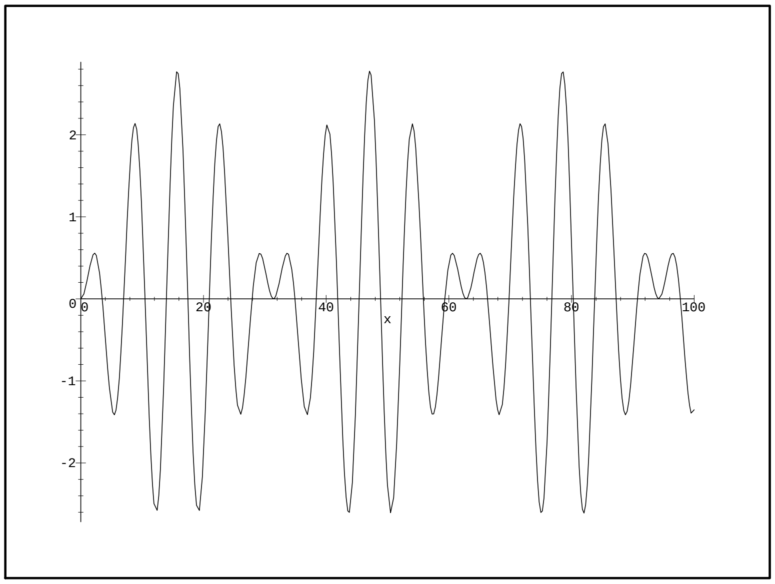


Figure A.8: The graph for assignment 2, problem 4

```
soln4:=dsolve({ode4,y(0)=0,D(y)(0)=0},y(x));
```

$$soln4 := \mathrm{y}(x) = \frac{100}{9}\cos\left(\frac{1}{5}x\right)^4 - \frac{100}{9}\cos\left(\frac{1}{5}x\right)^2 + \frac{25}{18} - \frac{25}{18}\cos(x)$$

```
plot(rhs(soln4),x=0..100);
```

A.2.5 Problem 5: Resonance and a MAPLE error

```
ode5:=diff(y(x),x$2)+y(x)=cos(x)/2;
```

$$ode5 := \left(\frac{\partial^2}{\partial x^2}\mathrm{y}(x)\right) + \mathrm{y}(x) = \frac{1}{2}\cos(x)$$

```
soln5:=dsolve({ode5,y(0)=0,D(y)(0)=0},y(x));
```

$$soln5 := \mathrm{y}(x) = \frac{1}{2}\sin(x)\arctan\left(\frac{\sin(x)}{\cos(x)+1}\right)$$

```
plot(rhs(soln5),x=0..40);
```

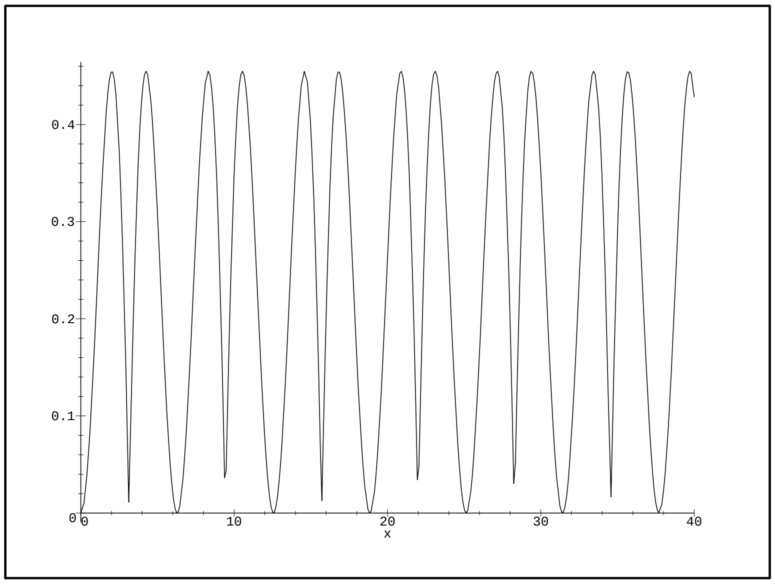


Figure A.9: The graph for assignment 2, problem 5.

It is not easy to see that the graph in Figure A.9 is incorrect. One cause for concern is that one expects a twice differentiable function for the solution, whereas Figure A.9 has corners. But this might be due to the graphics rather than the presence of real corners. Also, the solution

$$\mathrm{y}(\,x\,) = \frac{1}{2}\sin(\,x\,)\arctan\left(\frac{\sin(\,x\,)}{\cos(\,x\,)+1}\right)$$

should not need an arc tangent function (as we know from undetermined coefficients). In practice it is difficult to spot MAPLE's errors unless one tries a number of different methods of solutions and compares the results to make sure they agree. In this case using a numerical solver gives the "correct" answer:

```
soln5a:=dsolve({ode5,y(0)=0,D(y)(0)=0},y(x),type=numeric);

soln5a := proc(rkf45_x) ... end

odeplot(soln5a,[x,y(x)],0..40,numpoints=100);
```

A.2.6 Problem 6: Long term response.

```
ode6:=diff(y(x),x$2)+diff(y(x),x)/8+y(x)=3*cos(2*x);
```

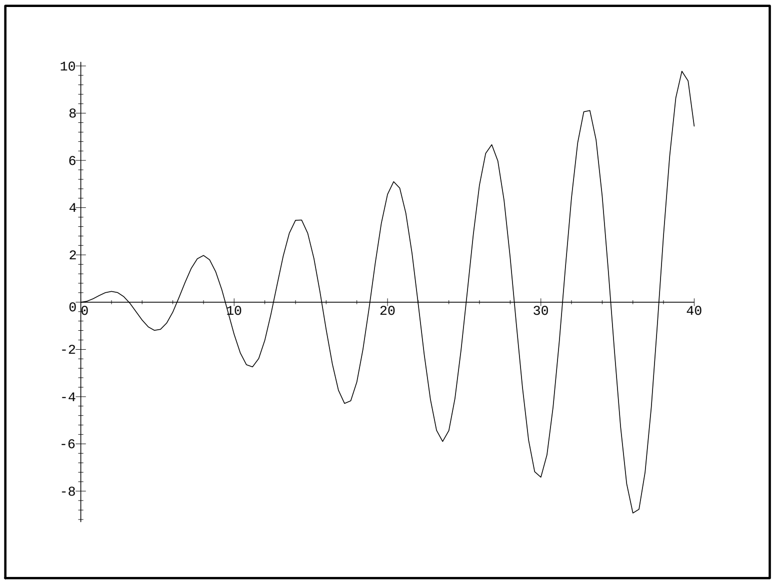


Figure A.10: The graph for assignment 2, problem 5 (correct picture!).

$$ode6 := \left(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(\,x\,)\right) + \frac{1}{8}\,\left(\frac{\partial}{\partial x}\,\mathrm{y}(\,x\,)\right) + \mathrm{y}(\,x\,) = 3\cos(\,2\,x\,)$$

```
soln6:=simplify(expand(dsolve({ode6,y(0)=0,D(y)(0)=0},y(x))));
```

$$soln6 := \mathrm{y}(\,x\,) = -\frac{16}{2465}\,\sqrt{255}\,\mathrm{e}^{(\,-1/16\,x\,)}\sin\left(\frac{1}{16}\,\sqrt{255}\,x\right) - \frac{288}{145}\cos(\,x\,)^2 + \frac{144}{145} + \frac{144}{145}\,\mathrm{e}^{(\,-1/16\,x\,)}\cos\left(\frac{1}{16}\,\sqrt{255}\,x\right) + \frac{24}{145}\sin(\,x\,)\cos(\,x\,)$$

```
plot(rhs(soln6),x=0..50);
```

A.3 Assignment 3: Inhomogeneous problems.

A.3.1 Problem 1a): Undetermined Coefficients.

First we solve by undetermined coefficients along the lines of the hand-out on this:

```
with(plots):with(DEtools):
ode1:=diff(y(x),x$2)+25*y(x)=sin(11*x/2);
```

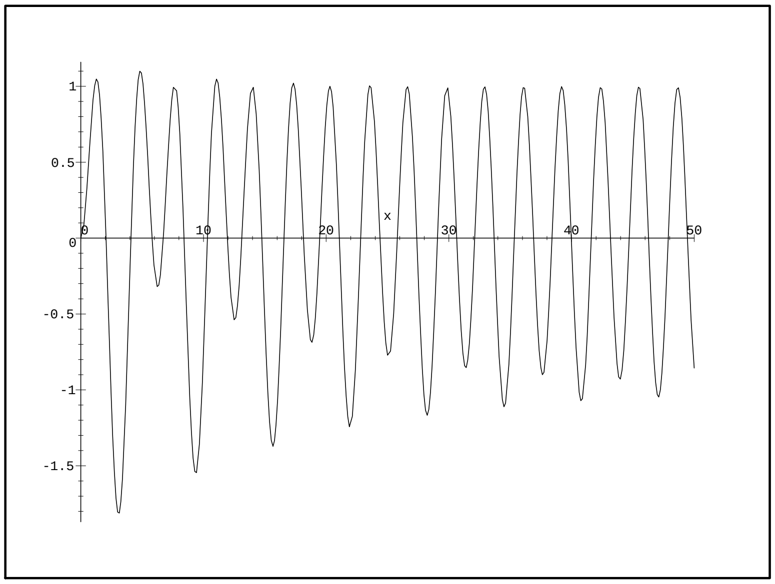


Figure A.11: The graph for assignment 2, problem 6.

```
Yc:=dsolve(lhs(ode1)=0,y(x));
Yp:=A*cos(11*x/2)+B*sin(11*x/2);
eq1:=simplify(subs(y(x)=Yp,ode1));
eq1:=lhs(eq1)-rhs(eq1)=0;
solve({coeff(lhs(eq1),cos(11*x/2)),coeff(lhs(eq1),sin(11*x/2))},{A,B});
assign(");
gs1:=rhs(Yc)+Yp;
eq1:=simplify(subs(x=0,gs1))=0;
eq2:=simplify(subs(x=0,diff(gs1,x)))=0;
solve({eq1,eq2},{_C1,_C2});
assign(");
plot(gs1,x=0..30);
```

A plot of the solution is shown in Figure A.12. This is a typical example of beating.

A.3.2 Problem 1b): Variation of Parameters.

Next we solve the same problem by variation of parameters:

```
restart;
with(plots):with(DEtools):
ode1:=diff(y(x),x$2)+25*y(x)=sin(11*x/2);
```

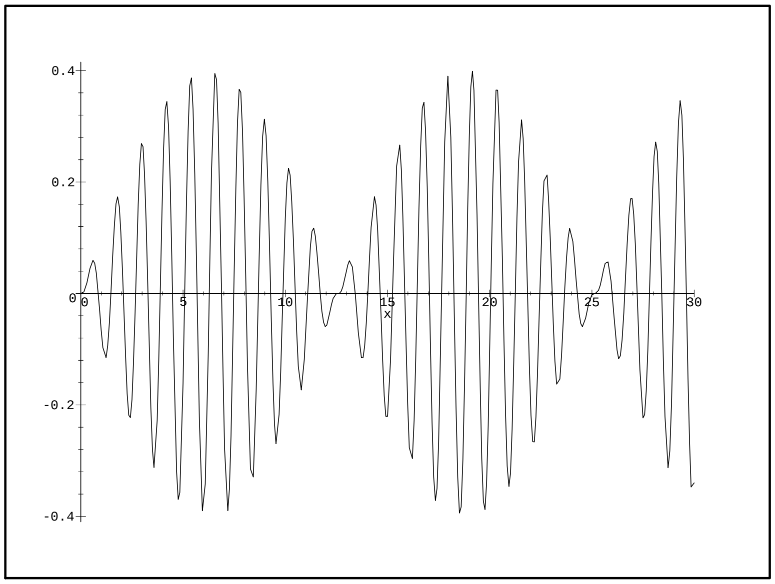


Figure A.12: The graph for assignment 3, problem 1

```
Yc:=dsolve(lhs(ode1)=0,y(x));
y1:=rhs(subs(_C1=1,_C2=0,Yc));
y2:=rhs(subs(_C1=0,_C2=1,Yc));
g:=rhs(ode1);
with(linalg):
Wron:=simplify(det(Wronskian([y1,y2],x)));
Yp:=simplify(-y1*int(simplify(y2*g/Wron),x)+y2*int(simplify(y1*g/Wron),x),trig);
gs2:=rhs(Yc)+Yp;
eq1:=simplify(subs(x=0,gs2))=0;
eq2:=simplify(subs(x=0,diff(gs2,x)))=0;
solve({eq1,eq2},{_C1,_C2});
assign(");
plot(gs2,x=0..30);
```

Although the solution looks rather messy, the plot is exactly the same as Figure A.12.

A.3.3 Problem 1c): Direct use of `dsolve`.

Finally, we solve the same problem by `dsolve` directly:

```
restart;
with(plots);with(DEtools);
```

```
ode1:=diff(y(x),x$2)+25*y(x)=sin(11*x/2);
soln:=simplify(dsolve({ode1,y(0)=0,D(y)(0)=0},y(x)),trig);
plot(rhs(soln),x=0..30);
```

Again the solution is messy, but the plot is identical to Figure A.12.

These example illustrate the fact that solutions obtained in MAPLE may not have the simple form you expect. Only by using undetermined cofficients can you be sure that you will get the simple looking solution.

A.3.4 Problem 2: Resonance.

This question asks you to look at the long term solution of a problem. This part of the solution is the particular solution (with solutions of the homogeneous problem removed). The previous question showed that you can get the simplest form of the solution using undetermined coefficients, so we will use that technique here. First we find the general solution:

```
restart;
with(plots):with(DEtools):
ode:=diff(y(x),x$2)+diff(y(x),x)+25*y(x)=10*sin(omega*x);
Yc:=dsolve(lhs(ode)=0,y(x));
Yp:=A*cos(omega*x)+B*sin(omega*x);
eqn:=simplify(subs(y(x)=Yp,ode));
eqn:=lhs(eqn)-rhs(eqn);
solve({coeff(eqn,cos(omega*x)),coeff(eqn,sin(omega*x))},{A,B});
assign(");
gs:=rhs(Yc)+Yp;
eq1:=simplify(subs(x=0,gs))=0;
eq2:=simplify(subs(x=0,diff(gs,x)))=0;
solve({eq1,eq2},{_C1,_C2});
assign(");
gs;
```

This gives the following solution to the problem:

$$
\begin{aligned}
y(x) = {} & 10\,\frac{\omega\, e^{(-1/2\,x)} \cos\left(\frac{3}{2}\sqrt{11}\,x\right)}{\%1} \\
& + \frac{10}{33}\,\frac{\sqrt{11}\,\omega\,(2\,\omega^2 - 49)\, e^{(-1/2\,x)} \sin\left(\frac{3}{2}\sqrt{11}\,x\right)}{\%1} - 10\,\frac{\omega\cos(\omega\, x)}{\%1} \\
& - 10\,\frac{(\omega^2 - 25)\,\sin(\omega\, x)}{\%1} \\
\%1 := {} & -49\,\omega^2 + \omega^4 + 625
\end{aligned}
$$

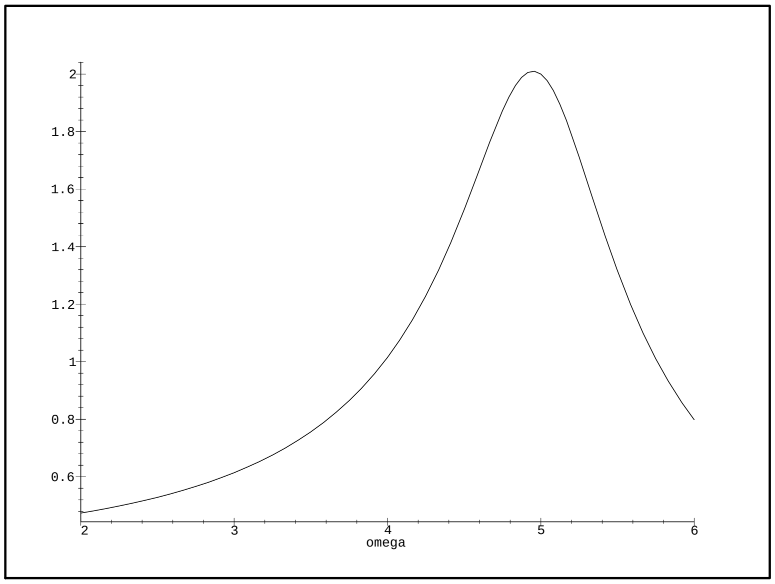


Figure A.13: A graph for assignment 3, problem 2. This shows the amplitude R of the long term response as a function of ω.

The long term solution is

$$y_p(x) = -10\,\frac{\omega\cos(\,\omega\,x\,)}{-49\,\omega^2+\omega^4+625} - 10\,\frac{(\omega^2-25)\,\sin(\,\omega\,x\,)}{-49\,\omega^2+\omega^4+625}$$

The amplitude R is given by $R=\sqrt{A^2+B^2}$ which we compute next. Then we find where $dR/d\omega=0$ and graph R as a function of ω;

```
R:=sqrt(A^2+B^2);
solve(diff(R,omega)=0,omega);
plot(R,omega=2..6);
```

The graph of R as a function of ω is shown in Figure A.13. Clearly, R has a maximum close to $\omega=5$ and the results of the derivative computation show that the optimal value is

$$\omega=\sqrt{\frac{49}{2}}\approx 4.949747\cdots$$

Next we compute the optimal amplitude, and the solution corresponding to this choice of ω. Then we graph the solution:

```
evalf(subs(omega=sqrt(49/2),R));
gs:=subs(omega=sqrt(49/2),gs);
plot(gs,x=0..20,numpoints=1000);
```

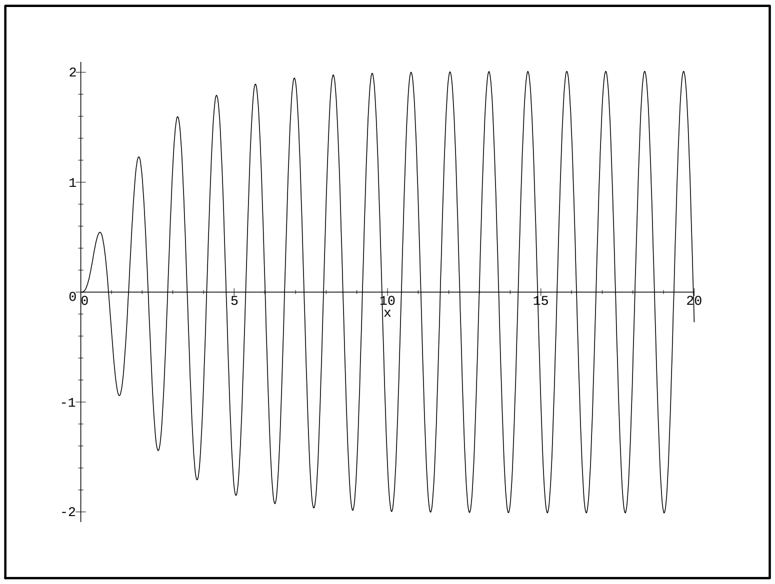


Figure A.14: A graph for assignment 3, problem 2. This shows the long term response as a function of t for the optimal value of ω.

The plot shown in Figure A.14 confirms that the solution has an optimal amplitude of

$$R_{max} = \frac{20}{33}\sqrt{11} \approx 2.0101\cdots$$

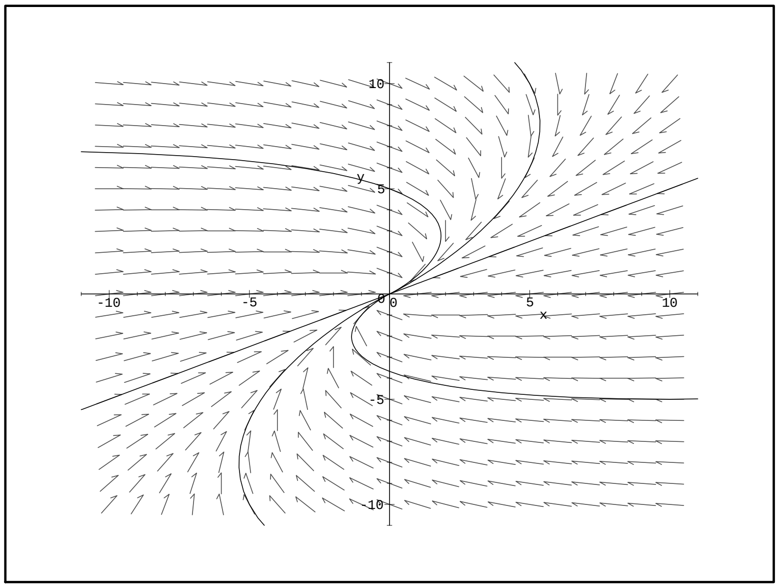


Figure A.15: Phase plane and direction field for assignment number four, problem 1

A.4 Assignment 4 - Systems of First Order Linear Equations

1. The MAPLE commands for this question are given next. Here there is no need to compute the eigenvectors for both eigenvalues since the eigenvalues are repeated:

```
with(DEtools):with(linalg):
alias(Id=&*());
A:=matrix(2,2,[[-3/2,1],[-1/4,-1/2]]);
evs:=eigenvals(A);
sp1:=nullspace(A-evs[1]*Id);
DEplot2(A,[x,y],t=-10..10,{[0,0,5],[0,10,-5],[0,2,1],[0,5,10],[0,-5,-10],
[0,-2,-1]},x=-10..10,y=-10..10,stepsize=0.1);
```

   MAPLE reports that the eigenvalues are $\lambda_1 = \lambda_2 = -1$ and the eigenvector is $\boldsymbol{v}^{(1)} = (2,1)^T$. The equilibrium point is an asymptotically stable improper node. The phase plane and direction field (the output from `DEplot2`) is shown in Figure A.15.

2. The MAPLE commands for this question are given next. Here there is no need to compute the eigenvectors for both eigenvalues since the eigenvectors are complex conjugates of oneanother.

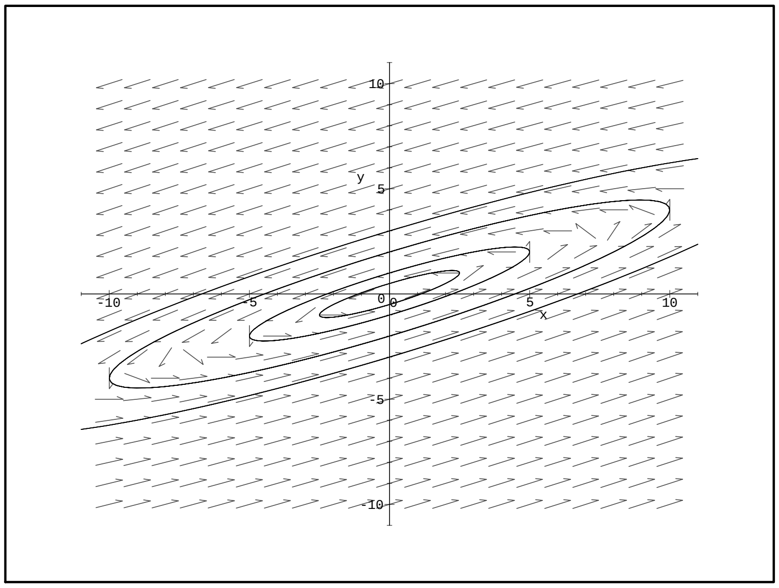


Figure A.16: Phase plane and direction field for assignment number four, problem 2

```
with(DEtools):with(linalg):
alias(Id=&*());
A:=matrix(2,2,[[2,-5],[1,-2]]);
evs:=eigenvals(A);
sp1:=nullspace(A-evs[1]*Id);
DEplot2(A,[x,y],t=-10..10,{[0,0,1],[0,0,2],[0,0,1/2],[0,0,3]},
x=-10..10,y=-10..10,stepsize=0.1);
```

MAPLE reports that the eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$ and the eigenvectors are $\boldsymbol{v}^{(1)} = (2-i,1)^T$ and $\boldsymbol{v}^{(2)} = (2+i,1)^T$. The equilibrium point is an stable center. The phase plane and direction field (the output from `DEplot2`) is shown in Figure A.16.

3. The MAPLE commands for this question are given next. Here there is no need to compute the eigenvectors for both eigenvalues since the eigenvectors are complex conjugates of oneanother.

```
with(DEtools):with(linalg):
alias(Id=&*());
A:=matrix(2,2,[[3,-2],[4,-1]]);
evs:=eigenvals(A);
sp1:=nullspace(A-evs[1]*Id);
DEplot2(A,[x,y],t=-10..10,{[0,0,1],[0,0,2],[0,0,1/2],[0,0,3]},x=-10..10,
```

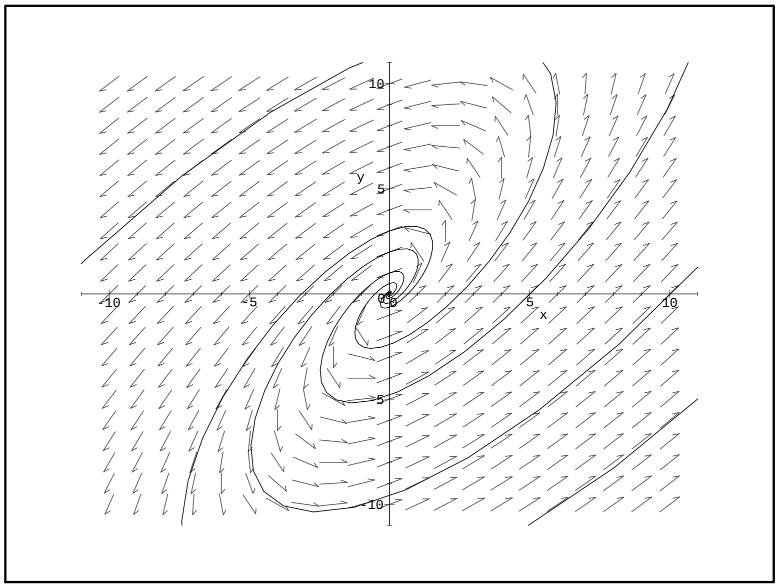


Figure A.17: Phase plane and direction field for assignment number four, problem 3

```
y=-10..10,stepsize=0.1);
```

MAPLE reports that the eigenvalues are $\lambda_1 = 1+2i$ and $\lambda_2 = 1-2i$ and the eigenvectors are $\boldsymbol{v}^{(1)} = (1, 1-i)^T$ and $\boldsymbol{v}^{(2)} = (1, 1+i)^T$. The equilibrium point is an unstable spiral point. The phase plane and direction field (the output from `DEplot2`) is shown in Figure A.17.

4. The MAPLE commands for this question are given next. Here there is we do need to compute the both eigenvectors.

```
with(DEtools):with(linalg):
alias(Id=&*());
A:=matrix(2,2,[[3,-2],[2,-2]]);
evs:=eigenvals(A);
sp1:=nullspace(A-evs[1]*Id);
sp2:=nullspace(A-evs[2]*Id);
DEplot2(A,[x,y],t=-10..10,{[0,1,2],[0,-1,-2],[0,2,1],[0,-2,-1],[0,-10,-5.5],
[0,-10,-3],[0,10,3],[0,10,5.5]},x=-10..10,y=-10..10,stepsize=0.1);
```

MAPLE reports that the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$ and the eigenvectors are $\boldsymbol{v}^{(1)} = (2, 1)^T$ and $\boldsymbol{v}^{(2)} = (1, 2)^T$. The equilibrium point is an unstable saddle point point. The phase plane and direction field (the output from `DEplot2`) is shown in Figure A.18.

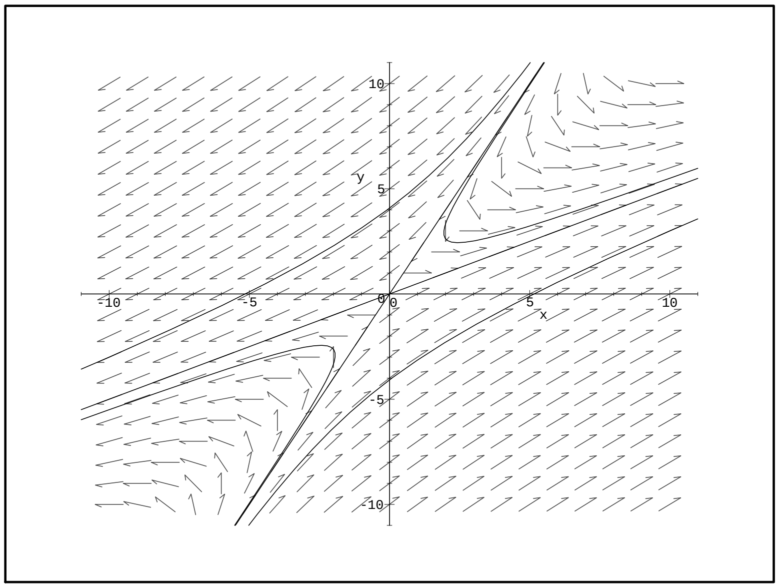


Figure A.18: Phase plane and direction field for assignment number four, problem 4

5. The MAPLE commands for this question are given next. Here there is we do need to compute the both eigenvectors.

```
with(DEtools):with(linalg):
alias(Id=&*());
A:=matrix(2,2,[[1,-2],[3,-4]]);
evs:=eigenvals(A);
sp1:=nullspace(A-evs[1]*Id);
sp2:=nullspace(A-evs[2]*Id);
DEplot2(A,[x,y],t=-10..10,{[0,1,1],[0,-1,-1],[0,2,3],[0,-2,-3],[0,-10,-5.5],
[0,10,5.5],[0,0,8],[0,0,-8]},x=-10..10,y=-10..10,stepsize=0.1);
```

MAPLE reports that the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -2$ and the eigenvectors are $\boldsymbol{v}^{(1)} = (1,1)^T$ and $\boldsymbol{v}^{(2)} = (-2,1)^T$. The equilibrium point is an asymptotically stable improper node. The phase plane and direction field (the output from `DEplot2`) is shown in Figure A.19.

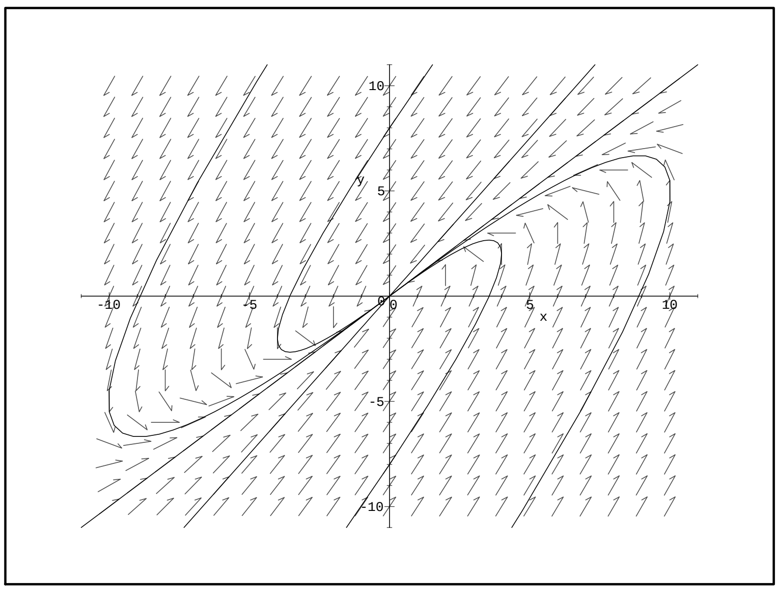


Figure A.19: Phase plane and direction field for assignment number four, problem 5

Appendix B

Answers to a few problems from Boyce and DiPrima

B.1 Chapter 3 - Second Order Problems

Page 142. Number 19. The characteristic equation is $r^2 - 2r + 5 = 0$ and it's roots are $r_1 = 1 + 2i$ and $r_2 = 1 - 2i$. Thus the general soltion is

$$y(x) = c_1 \exp(x) \cos(2x) + c_2 \exp(x) \sin(2x).$$

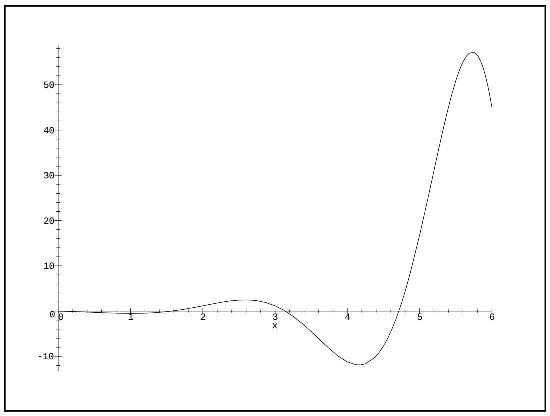
To find the constants, we use the initial conditions:

$$y\left(\frac{\pi}{2}\right) = c_1 \exp\left(\frac{\pi}{2}\right) \cos(\pi) + c_2 \exp\left(\frac{\pi}{2}\right) \sin(\pi) = -c_1 \exp\left(\frac{\pi}{2}\right) = 0.$$

so $c_1 = 0$. Using this fact,

$$y'\left(\frac{\pi}{2}\right) = c_2 \exp\left(\frac{\pi}{2}\right) \sin(\pi) + 2c_2 \exp\left(\frac{\pi}{2}\right) \cos(\pi) = 2$$

So $c_2 = -\exp(-\pi/2)$ and $y(x) = -\exp(x - \pi/2)\sin(2x)$. The graph looks like (a growing oscillation):



Page 150. Number 19. Use the method of reduction of order. Let $y(x) = xv(x)$, then $y'(x) = v + xv'$ and $y''(x) = 2v' + xv''$. Substituting into the differential equation we find

$$xv'' + 4v' = 0$$

So if $u(x) = v'(x)$ we have $xu' + 4u = 0$ and so $u(x) = -3x^{-4}$ (I have made a particular choice of the constant of integration to get the -3). Then $v' = -3x^{-4}$ so $v(x) = x^{-3}$ and hence $y_2(x) = xx^{-3} = x^{-2}$.

Page 162. Number 14. We seek two particular solutions. The first denoted y_{1p} is for $y_1'' + 4y_1 = x^2$ and the second (y_{2p}) is for $y_2'' + 4y_2 = 3\exp(x)$. Then a particular solution for the full problem is $y_p = y_{1p} + y_{2p}$.

The characteristic equation is $r^2 + 4 = 0$ so $r_1 = 2i$ and $r_2 = -2i$. For y_{1p} we guess

$$y_{1p}(x) = Ax^2 + Bx + C$$

And substituting this into the equation for y_1 yields $A = 1/4$, $B = 0$ and $C = -1/8$. For y_{2p} we guess

$$y_{2p}(x) = D\exp(x)$$

and find that $D = -3/5$. Hence

$$y_p(x) = \frac{1}{4}x^2 - \frac{1}{8} + \frac{3}{5}\exp(x)$$

The general solution is

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x) + \frac{1}{4}x^2 - \frac{1}{8} + \frac{3}{5}\exp(x).$$

From the initial condition $y(0) = 0$ we get $c_1 = -3/5 + 1/8 = -19/40$ and from $y'(0) = 2$ we obtain $c_2 = 7/10$. Hence the solution is

$$y(x) = -\frac{19}{40}\cos(2x) + \frac{7}{10}\sin(2x) + \frac{1}{4}x^2 - \frac{1}{8} + \frac{3}{5}\exp(x).$$

Page 169. Number 14. Substitute $y_1(x) = x$ and $y_2(x) = x\exp(x)$ into the differential equation to show they satisfy the differential equation. To get y_p we compute

$$W[y_1, y_2] = \det\begin{pmatrix} x & x\exp(x) \\ 1 & \exp(x) + x\exp(x) \end{pmatrix} = x^2\exp(x) \neq 0 \text{ if } x > 0$$

hence $\{x, x\exp(x)\}$ is a fundamental set. Using variation of parameters with the above choice of y_1 and y_2 and

$$g(x) = 2x^3/x^2 = 2x$$

gives

$$y_p(x) = xu_1(x) + x\exp(x)u_2(x)$$

where

$$u_1(x) = -\int^x \frac{x\exp(x)2x}{x^2\exp(x)}\,dx = -2x \text{ and } u_2(x) = \int^x \frac{x2x}{x^2\exp(x)}\,dx = -2\exp(-x).$$

so

$$y_p(x) = -2x^2 - 2x$$

Since $2x$ is a solution of the homogeneous problem ($2x = 2y_1(x)$) we can simplify the particular solution to $y_p(x) = -2x^2$.

Page 181. Number 9. $m = 20$g, $\gamma = 400$dyne-sec/cm, $k = 20 * 980/5 = 3920$dyne/cm. So the equation is $10u'' + 400u' + 3920u = 0$ or $u'' + 20u' + 196u = 0$. Characteristic equation is $r^2 + 20r + 196 = 0$ so $r_1 = -10 + 4\sqrt{6}i$ and $r_2 = -10 - 4\sqrt{6}i$. Hence

$$u(t) = c_1\exp(-10t)\cos(4\sqrt{6}t) + c_2\exp(-10t)\sin(4\sqrt{6}t)$$

The initial conditions are $u(0) = 2$ and $u'(0) = 0$ so $c_1 = 2$ and $c_2 = 5\sqrt{6}/6$. The solution is

$$u(t) = 2\exp(-10t)\cos(4\sqrt{6}t) + \frac{5\sqrt{6}}{6}\exp(-10t)\sin(4\sqrt{6}t)$$

The quasi-frequency is $\mu = 4\sqrt{6}$ and the quasi-period is $T_d = \pi/(2\sqrt{6})$. Hence $T_d/T = 7\sqrt{6}/12$.

Page 189. Number 11. $m = 8/32\text{lb-sec}^2/\text{ft}$. Damping constant $\gamma = 0.25\text{lb-sec/ft}$. $k = 8/(1/2) = 16\text{lb/ft}$. $F(t) = 4\cos(2t)\text{lb}$. To find the steady state response, we need to find a particular solution of the following ODE

$$u'' + u' + 64u = 16\cos(2t)$$

Suppose $u_p(t) = A\cos(2t) + B\sin(2t)$ then substituting into the equation we have

$$(60A + 2B)\cos(2t) + (60B - 2A)\sin(2t) = 16\cos(2t)$$

So that $A = 240/901$ and $B = 8/901$. Hence

$$u_p(t) = \frac{240}{901}\cos(2t) + \frac{8}{901}\sin(2t)$$

This is the long term response of the system since the transient response (solution of the homogeneous problem) dies away due to damping.

The amplitude of the long term response is

$$R = \frac{F_0}{\Delta} = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

The maximum amplitude occurs when Δ^2 is a minimum. But $\Delta^2 = m^2(k/m - \omega^2)^2 + \gamma^2\omega^2$ so

$$\frac{d\Delta^2}{dm} = 0 \text{ implies that } m = k/\omega^2$$

so $m = 16/2^2 = 4$ slugs.

B.2 Chapter 5 - Series Solution

Page 227. Number 16. Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \text{ so that } y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \text{ and } y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Substituting into the differential equation:

$$(2 + x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

Mutiplying through by the factors outside the sum signs:

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

Changing index of summation to obtain x^n in each term:

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

Adjusting limits of summation:

$$4a_2+12a_3x-a_1x+4a_0+4a_1x+\sum_{n=2}^{\infty}\left(2(n+2)(n+1)a_{n+2}+n(n-1)a_n-na_n+4a_n\right)x^n=0$$

Hence we obtain:

$$a_2 = a_0, \quad a_3 = -a_1/4, \text{ and } a_{n+2} = -\frac{n^2-2n+4}{2(n+2)(n+1)}a_n,\ n > 3.$$

Hence the general solution is

$$y(x) = a_0\left(1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + \cdots\right) + a_1\left(x - \frac{x^3}{4} + \frac{7x^5}{160} + \cdots\right)$$

Then $y(0) = -1$ implies $a_0 = -1$ and $y'(0) = 3$ implies that $a_1 = 3$ so

$$y(x) = -1 + 3x + x^2 - \frac{3}{4}x^3 - \frac{1}{6}x^4 + \frac{1}{160}x^5 + \cdots.$$

B.3 Chapter 6 - Laplace Transforms

Page 303. Number 10. Let $Y(s) = \mathcal{L}(y(t))$ then

$$(s^2 + s + 5/4)Y = \mathcal{L}(g(t))$$

But $g(t) = \sin(t) - u_\pi(t)\sin(t) = \sin(t) + u_\pi(t)\sin(t-\pi)$ and hence

$$\mathcal{L}(g(t)) = \frac{1}{s^2+1} + \exp(-\pi s)\frac{1}{s^2+1}$$

Hence

$$Y(s) = \frac{1}{s^2+s+5/4}\frac{1}{s^2+1}\left(1+\exp(-\pi s)\right)$$

Then to find $\mathcal{L}^{-1}Y$ we use partial fractions:

$$\frac{1}{(s^2+s+5/4)(s^2+1)} = \frac{As+B}{s^2+s+5/4} + \frac{Cs+D}{s^2+1}$$

so that

$$1 = (As+B)(s^2+1) + (Cs+D)(s^2+s+5/4)$$

which requires that

$$A + C = 0, \quad B + D + C = 0, \quad D + A + 5C/4 = 0 \text{ and } B + 5D/4 = 1$$

The solution is

$$A = 16/17, \quad B = 12/17, \quad C = -16/17 \text{ and } D = 4/17$$

But $s^2 + s + 5/4 = (s + 1/2)^2 + 1$ so we write

$$\frac{1}{(s^2 + s + 5/4)(s^2 + 1)} = \frac{16(s + 1/2)/17 + 4/17}{(s + 1/2) + 1} + \frac{-16s/17 + 4/17}{s^2 + 1}$$

Thus

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2 + s + 5/4)(s^2 + 1)}\right) = \frac{16}{17}\exp(-t/2)\cos(t) + \frac{4}{17}\exp(-t/2)\sin(t) + -\frac{16}{17}\cos(t) + \frac{4}{17}\sin(t)$$

If we let

$$h(t) = \frac{16}{17}\exp(-t/2)\cos(t) + \frac{4}{17}\exp(-t/2)\sin(t) + -\frac{16}{17}\cos(t) + \frac{4}{17}\sin(t)$$

then

$$y(t) = h(t) + u_\pi(t)h(t - \pi)$$

Page 297. Number 25. Write f using step functions as

$$f(t) = 1 - u_1(t) + u_2(t) - u_3(t)$$

then

$$\begin{aligned} \mathcal{L}(f(t)) &= \mathcal{L}(1) - \mathcal{L}(u_1(t)) + \mathcal{L}(u_2(t)) - \mathcal{L}(u_3(t)) \\ &= \frac{1}{s}(1 - \exp(-s) + \exp(-2s) - \exp(-3s)) \text{ for } s > 0. \end{aligned}$$

Page 303. Number 9. Write $g(t)$ using step functions as

$$g(t) = t - u_1(t)(t - 1)$$

So of $Y = \mathcal{L}(y(t))$ we have (taking Laplace transforms of the differential equation:

$$(s^2 + 1)Y - 1 = \mathcal{L}(t - u_1(t)(t - 1)) = \frac{1 - \exp(-s)}{s^2}$$

Hence

$$Y(s) = \frac{1}{s^2 + 1}\left(\frac{1 - \exp(-s)}{s^2} + 1\right) = \frac{1}{s^2} + \frac{\exp(-s)}{s^2 + 1} - \frac{\exp(-s)}{s^2}$$

From Table 6.2.1 in your book

$$y(t) = \mathcal{L}^{-1}(Y(s)) = t - u_1(t)(t - 1 - sin(t - 1))$$

Page 313. Number 15. Let $Y(s) = \mathcal{L}(y(t))$ then

$$s^2Y - s + 1 + sY - 1 + \frac{5}{4}Y = \frac{1 - \exp - \pi s}{s}$$

So

$$\begin{aligned} Y &= \frac{s}{s^2 + s + 5/4} + \frac{1 - \exp(-\pi s)}{s(s^2 + s + 5/4)} \\ &= \frac{(s + 1/2) - 1/2}{(s + 1/2)^2 + 1} + \frac{1 - \exp(-\pi s)}{s} \frac{1}{(s + 1/2)^2 + 1} \end{aligned}$$

But

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{(s + 1/2)}{(s + 1/2)^2 + 1}\right) &= \exp(-t/2)\cos(t) \\ \mathcal{L}^{-1}\left(\frac{-1/2}{(s + 1/2)^2 + 1}\right) &= -\frac{1}{2}\exp(-t/2)\sin(t) \\ \mathcal{L}^{-1}\left(\frac{1 - \exp(-\pi s)}{s}\right) &= 1 - u_\pi(t) \\ \mathcal{L}^{-1}\left(\frac{1}{(s + 1/2)^2 + 1}\right) &= \exp(-t/2)\sin(t) \end{aligned}$$

So by the convolution theorem

$$y(t) = \exp(-t/2)\cos(t) - \frac{1}{2}\exp(-t/2)\sin(t) + \int_0^t \exp(-(t-\tau)/2)\sin(t-\tau)(1-u_\pi(\tau)\, d\tau$$

B.4 Chapter 7 - Systems of First Order Equations

Page 344. Number 21. Find the eigenvalues from the characteristic equation:

$$det(A - \lambda I) = (1 - \lambda)(\lambda^2 - 2\lambda + 5) = 0$$

Hence $\lambda_1 = 1$, $\lambda_2 = 1 + 2i$ and $\lambda_3 = 1 - 2i$.

Now we find the eigenvectors using $(A - \lambda I)\boldsymbol{x} = 0$.
First $\lambda_1 = 1$ then

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus $x_1 = x_3$ and $x_1 = -2x_2/3$. We can choose $x_2 = 3$, then $x_1 = -2$ and $x_3 = -2$ so

$$\boldsymbol{x}^{(1)} = \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix}$$

Next $\lambda_2 = 1 + 2i$ then

$$\begin{pmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus $x_1 = 0$ and $-ix_2 = x_3$. We can choose $x_2 = 1$, then $x_3 = -i$ so

$$\boldsymbol{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}$$

Finally $\lambda_3 = 1 - 2i$ then

$$\begin{pmatrix} 2i & 0 & 0 \\ 2 & 2i & -2 \\ 3 & 2 & 2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus $x_1 = 0$ and $ix_2 = x_3$. We can choose $x_2 = 1$, then $x_3 = i$ so

$$\boldsymbol{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$$

Page 349. Number 6. a) The Wronskian W is given by

$$W[\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}] = \det \begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix} = 2t^2 - t^2 = t^2$$

b) Since W vanishes only at $t = 0$ we know that $\{\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}\}$ is linearly independent for $-\infty < t < 0$ or $0 < t < \infty$.
c) If the differential equation is

$$\boldsymbol{x}' = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \boldsymbol{x}$$

we know that $W' = (a_{1,1} + a_{2,2})W$ so

$$a_{1,1} + a_{2,2} = 2t/t^2 = 2/t$$

Hence the coefficient matrix is discontinuous at $t = 0$.
d) Using

$$\boldsymbol{x}' = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \boldsymbol{x}$$

we know that

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} t^2 \\ 2t \end{pmatrix} = \begin{pmatrix} 2t \\ 2 \end{pmatrix}$$

Thus

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} 1 & 2t \\ 0 & 2 \end{pmatrix} \begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix}^{-1}$$

and so

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2/t^2 & 2/t \end{pmatrix}$$

Page 357. Number 1. First find the eigenvalues

$$\det \begin{pmatrix} -2-\lambda & 1 \\ -5 & 4-\lambda \end{pmatrix} = 0 \text{ so } \lambda^2 + 2\lambda - 3 = 0 \text{ and } \lambda_1 = 3,\ \lambda_2 = -1.$$

Now find the eigenvectors. First $\lambda = \lambda_1 = 3$

$$\begin{pmatrix} -5 & 1 \\ -5 & 1\lambda \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so $\boldsymbol{\xi}^(1) = (1, 5)^T$. Next the eigenvector for $\lambda = \lambda_2 = -1$

$$\begin{pmatrix} -1 & 1 \\ -5 & 5\lambda \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

s $\boldsymbol{\xi}^{(2)} = (1, 1,)^T$. Thus the general solution is

$$\boldsymbol{x}(t) = c_1 \exp(3t) \begin{pmatrix} 1 \\ 5 \end{pmatrix} + c_2 \exp(-t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Because $\boldsymbol{x}(0) = (1, 3)^T$

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and $c_1 = c_2 = 1/2$. The solution is

$$\boldsymbol{x}(t) = \frac{1}{2} \exp(3t) \begin{pmatrix} 1 \\ 5 \end{pmatrix} + \frac{1}{2} \exp(-t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

As $t \to \infty$, the solution approaches $x_2 = 5x_1$ an is unbounded.

Page 369. Number 9. First find the eigenvalues

$$\det\begin{pmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{pmatrix} = 0 \text{ so } \lambda^2 + 2\lambda + 2 = 0 \text{ and } \lambda_1 = 3 - 1 + i \; \lambda_2 = -1 - i.$$

Now find the eigenvectors. First $\lambda = \lambda_1 = -1 + i$

$$\begin{pmatrix} -2-i & -5 \\ 1 & -2-i\lambda \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so $\boldsymbol{\xi}^((1) = (2+i, 1)^T = (2,1)^T + i(1, 0^T$. There is no need to find the second eigenvector (it is the conjugate of the one we just found). The general solution is

$$\begin{aligned} \boldsymbol{x}(t) &= \exp(-t)\left[c_1\left(\cos(t)\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sin(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + c_2\left(\sin(t)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)\right] \\ &= \exp(-t)\left[c_1\begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix} + +c_2\begin{pmatrix} 2\sin(t) + \cos(t) \\ \sin(t) \end{pmatrix}\right] \end{aligned}$$

Because $\boldsymbol{x}(0) = (1,1)^T$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and $c_1 = 1$ and $c_2 = -1$. The solution is

$$\boldsymbol{x}(t) = \exp(-t)\begin{pmatrix} \cos(t) - 3\sin(t) \\ \cos(t) - \sin(t) \end{pmatrix}$$

The solution is an anti-clockwise spiral that approachs the origin as $t \to \infty$.

Page 439. Number 1. Determine the eigenvalues.

$$\det\begin{pmatrix} \lambda - 3 & 2 \\ -2 & \lambda + 2 \end{pmatrix} = \lambda^2 - \lambda - 2 = 0$$

So $\lambda_1 = -1$ and $\lambda_2 = 2$.

Now find the eigenvectors. First for $\lambda_1 = -1$.

$$\begin{pmatrix} -4 & 2 \\ -2 & 1 \end{pmatrix} = 0 \text{ so } \boldsymbol{v}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Next for $\lambda_2 = 2$.

$$\begin{pmatrix} -1 & 2 \\ -2 & 4 \end{pmatrix} = 0 \text{ so } \boldsymbol{v}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

So the general solution is

$$\boldsymbol{x}(t) = c_1 \exp(-t) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \exp(2t) \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The equilibrium point is an unstable saddle point.

Trajectories in the phase plane are shown in the next figure.