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Innermost Termination of Context-Sensitive Rewriting

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Abstract. Context-sensitive rewriting is a restriction of term rewriting used to model evaluation strategies in functional programming and in programming languages like OBJ. For example, under certain conditions termination of an OBJ program is equivalent to innermost termination of the corresponding context-sensitive rewrite system [25]. To prove termination of context-sensitive rewriting, several methods have been proposed in the literature which transform context-sensitive rewrite systems into ordinary rewrite systems such that termination of the transformed ordinary system implies termination of the original context-sensitive system. Most of these transformations are not very satisfactory when it comes to proving *innermost* termination. We investigate the relationship between termination and innermost termination of context-sensitive rewriting and we examine the applicability of the different transformations for innermost termination proofs. Finally, we present a simple transformation which is both sound and complete for innermost termination.

1 Introduction

Evaluation in functional languages is often guided by specific evaluation strategies. For example, in the program consisting of the rules

$$\mathsf{from}(x) \to x : \mathsf{from}(\mathsf{s}(x)) \qquad \mathsf{nth}(0, x : y) \to x \qquad \mathsf{nth}(\mathsf{s}(n), x : y) \to \mathsf{nth}(n, y)$$

a term like $\mathsf{nth}(\mathsf{s}(0), \mathsf{from}(0))$ admits a finite reduction to $\mathsf{s}(0)$ as well as infinite reductions. The infinite reductions can for instance be avoided by always contracting the outermost redex. Context-sensitive rewriting (Lucas [23, 24]) provides an alternative way of solving the non-termination problem and of dealing with infinite data objects. Rather than specifying which redexes may be contracted, in context-sensitive rewriting every n-ary function symbol f is equipped with a *replacement map* $\mu(f) \subseteq \{1, \ldots, n\}$ which indicates which arguments of f may be evaluated and a contraction of a redex is allowed only if it does not take place in a forbidden argument of a function symbol somewhere above it. So by defining $\mu(:) = \{1\}$, contractions in the argument t of a term $s : t$ are forbidden.

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Now in the example infinite reductions are no longer possible while normal forms can still be computed. (See [27] for the relationship between normalization under ordinary and under context-sensitive rewriting.) Context-sensitive rewriting can also model the usual evaluation strategy for conditionals.

Example 1.

$$
\begin{array}{rlr}
0 \leqslant y \to \mathsf{true} & & \mathsf{p}(0) \to 0 \\
\mathsf{s}(x) \leqslant 0 \to \mathsf{false} & & \mathsf{p}(\mathsf{s}(x)) \to x \\
\mathsf{s}(x) \leqslant \mathsf{s}(y) \to x \leqslant y & & \mathsf{if}(\mathsf{true}, x, y) \to x \\
x - y \to \mathsf{if}(x \leqslant y, 0, \mathsf{s}(\mathsf{p}(x) - y)) & & \mathsf{if}(\mathsf{false}, x, y) \to y
\end{array}
$$

Because of the rule for "$-$", this system is not terminating. However, in functional languages typically if's first argument is evaluated first and depending on the result either the second or third argument is evaluated afterwards. Again, this can easily be modeled with context-sensitive rewriting by the replacement map $\mu(\mathsf{if}) = \{1\}$ which forbids all reductions in the arguments t_2 and t_3 of $\mathsf{if}(t_1, t_2, t_3)$.

In programming languages like OBJ [6, 8, 16, 17], the user can supply strategy annotations to control the evaluation [9, 28, 29]. For every n-ary symbol f, a (positive) strategy annotation is a list $\varphi(f)$ of numbers $(i_1, \dots, i_k)$ from $\{0, 1, \dots, n\}$. When reducing a term $f(t_1, \dots, t_n)$ one first has to evaluate the i_1-th argument of f (if $i_1 > 0$), then one evaluates the i_2-th argument (if $i_2 > 0$), and so on, until a 0 is encountered. At this point one tries to evaluate the whole term $f(\dots)$ at its root position. So in order to enforce the desired evaluation strategy for if in Example 1, it has to be equipped with the strategy annotation $(1, 0)$.

Context-sensitive rewriting can simulate the evaluation strategy of OBJ. A strategy is called *elementary* if for every defined[1] symbol f, $\varphi(f)$ contains a single occurrence of 0, at the end. Lucas [25] showed that for elementary strategies, the OBJ program is terminating if and only if the corresponding context-sensitive rewrite system is *innermost* terminating.[2] Here $\mu(f)$ is defined to consist of all numbers greater than 0 in $\varphi(f)$. For example, the program with the rules

$$\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{a}) \qquad\qquad \mathsf{a} \to \mathsf{b}$$

is terminating if $\varphi(\mathsf{f}) = (1, 0)$ and $\varphi(\mathsf{a}) = (0)$. The corresponding context-sensitive system with $\mu(\mathsf{f}) = \{1\}$ is not terminating, but it is *innermost* terminating. Thus, to simulate OBJ evaluations with context-sensitive rewriting, we have to restrict ourselves to innermost reductions where (allowed) arguments to a function are evaluated before evaluating the function.

Because of this connection to OBJ programs and also because for rewrite systems innermost termination is easier to prove automatically than termination [1], it is worthwhile to investigate innermost termination of context-sensitive rewriting. (As an alternative approach, in [11] a method to prove termination

[1] Every symbol on the root position of a left-hand side of a rule is called *defined*. In Example 1 the defined symbols are "$\leqslant$", "$-$", p, and if. All remaining function symbols are called *constructors*.

[2] The "if" direction even holds without the restriction to elementary strategies [25].

of OBJ-like programs by direct induction proofs is proposed.) Termination of context-sensitive rewriting has been studied in a number of papers (e.g., [5, 10, 14, 15, 20, 23, 24, 27, 32]). Apart from a direct semantic characterization [32] and some recent extensions of standard termination methods for term rewriting to context-sensitive rewriting [5, 20], all other proposed methods transform context-sensitive rewrite systems (CSRSs) into ordinary term rewrite systems (TRSs) such that termination of the transformed TRS implies termination of the original CSRS (i.e., all these transformations are *sound*). Direct approaches to termination analysis of CSRSs and transformational approaches both have their advantages. Techniques for proving termination of ordinary term rewriting have been studied extensively (e.g., [21, 22, 7, 3, 30, 31, 1, 4]) and the main advantage of the transformational approach is that in this way, all termination techniques for ordinary TRSs including future developments can be used to infer termination of CSRSs. For instance, the methods of [5, 20] are unable to handle systems like Example 1. Of the five transformations described in [10, 14, 23, 32] only the second one of [14] is also *complete*: Termination of the original CSRS implies termination of the transformed TRS.

After introducing the termination problem of context-sensitive rewriting in Section 2, in Section 3 we review the results of Lucas [25] on innermost termination of context-sensitive rewriting and we show that the two transformations Θ_1 and Θ_2 of [14] are sound for innermost termination as well. Despite its soundness Θ_2 is not very useful for proving innermost termination, because termination and innermost termination coincide for the TRSs it produces. In Section 4 we show that for the class of orthogonal CSRSs, innermost termination already implies termination. This result is independent from the transformation framework and is of general interest when investigating the termination behavior of CSRSs. A consequence of this result is that for this particular class, Θ_1 is complete for innermost termination. In Section 5 we present a new transformation Θ_3 which is both sound and complete for innermost termination, for arbitrary CSRSs. Surprisingly, such a transformation can be obtained by just a small modification of Θ_1. In spite of the similarity between the two transformations, the new completeness proof is non-trivial. We make some remarks on a possible simplification of Θ_3 and on *ground* innermost termination in Section 6. In Section 7 we show that Θ_3 is equally powerful as Θ_1 when it comes to (non-innermost) termination. Finally, Appendix A demonstrates how innermost termination of the TRSs resulting from our new transformation is proved with *dependency pairs* [1].

2 Termination of Context-Sensitive Rewriting

Familiarity with the basics of term rewriting [2] is assumed. We require that every signature $\mathcal{F}$ contains a constant. A function $\mu\colon \mathcal{F} \to \mathcal{P}(\mathbb{N})$ is a *replacement map* if $\mu(f)$ is a subset of $\{1, \ldots, \mathrm{arity}(f)\}$ for all $f \in \mathcal{F}$. A *CSRS* $(\mathcal{R}, \mu)$ is a TRS $\mathcal{R}$ over a signature $\mathcal{F}$ equipped with a replacement map μ. The context-sensitive rewrite relation $\to_{\mathcal{R},\mu}$ is defined as the restriction of the usual rewrite relation

$\to_{\mathcal{R}}$ to contractions of redexes at *active* positions. A position π in a term t is active if $\pi = \epsilon$ (the root position), or $t = f(t_1, \dots, t_n)$, $\pi = i\pi'$, $i \in \mu(f)$, and π' is active in t_i. So $s \to_{\mathcal{R},\mu} t$ if and only if there is a rule $l \to r$ in $\mathcal{R}$, a substitution σ, and an active position π in s such that $s|_\pi = l\sigma$ and $t = s[r\sigma]_\pi$. If all active arguments of $l\sigma$ are in μ-normal form, then the reduction step is *innermost* and we write $s \xrightarrow{\mathsf{i}}_{\mathcal{R},\mu} t$. Here a μ-normal form is a normal form with respect to $\to_{\mathcal{R},\mu}$. We abbreviate $\to_{\mathcal{R},\mu}$ to $\to_\mu$ and $\xrightarrow{\mathsf{i}}_{\mathcal{R},\mu}$ to $\xrightarrow{\mathsf{i}}_\mu$ if $\mathcal{R}$ is clear from the context. A CSRS $(\mathcal{R}, \mu)$ is *left-linear* if the left-hand sides of the rewrite rules in $\mathcal{R}$ are linear terms (i.e., they do not contain multiple occurrences of the same variable). Let $l \to r$ and $l' \to r'$ be renamed versions of rewrite rules of $\mathcal{R}$ such that they have no variables in common and suppose $l|_\pi$ and l' are unifiable with most general unifier σ for some non-variable active position π in l. The pair of terms $\langle l[r']_\pi\sigma, r\sigma\rangle$ is a *critical pair* of $(\mathcal{R}, \mu)$, except when $l \to r$ and $l' \to r'$ are renamed versions of the same rewrite rule and $\pi = \epsilon$. A *non-overlapping* CSRS has no critical pairs and an *overlay* CSRS has no critical pairs with $\pi \neq \epsilon$. A CSRS is *orthogonal* if it is left-linear and non-overlapping. Notions like "termination" for a CSRS $(\mathcal{R}, \mu)$ always concern the relation $\to_\mu$ (i.e., they correspond to "μ-termination" in [24]).

To prove termination of CSRSs, several transformations from CSRSs to ordinary TRSs were suggested. We recall the transformations Θ_1 and Θ_2 of Giesl & Middeldorp and refer to [14, 15] for motivations. The main idea of Θ_1 is to use new unary symbols active and mark to indicate active positions in a term on the object level. If $l \to r$ is a rule in the CSRS then the transformed TRS contains the rule $\mathsf{active}(l) \to \mathsf{mark}(r)$. The symbol mark is used to traverse a term top-down in order to place the symbol active at all active positions.

Definition 2 (Θ_1). *Let $(\mathcal{R}, \mu)$ be a CSRS over a signature $\mathcal{F}$. The TRS $\mathcal{R}^1_\mu$ over the signature $\mathcal{F}_1 = \mathcal{F} \cup \{\mathsf{active}, \mathsf{mark}\}$ consists of the following rewrite rules:*

$$\begin{aligned} \mathsf{active}(l) &\to \mathsf{mark}(r) && \text{for all } l \to r \in \mathcal{R} \\ \mathsf{mark}(f(x_1, \dots, x_n)) &\to \mathsf{active}(f([x_1]^f_1, \dots, [x_n]^f_n)) && \text{for all } f \in \mathcal{F} \\ \mathsf{active}(x) &\to x \end{aligned}$$

Here $[t]^f_i = \mathsf{mark}(t)$ if $i \in \mu(f)$ and $[t]^f_i = t$ otherwise. We denote the subset of $\mathcal{R}^1_\mu$ consisting of all rules of the form

$$\mathsf{mark}(f(x_1, \dots, x_n)) \to \mathsf{active}(f([x_1]^f_1, \dots, [x_n]^f_n))$$

by $\mathcal{M}$. The transformation $(\mathcal{R}, \mu) \mapsto \mathcal{R}^1_\mu$ is denoted by Θ_1 and we shorten $\to_{\mathcal{R}^1_\mu}$ to $\to_1$.

Because every infinite reduction of a term t in the original CSRS would correspond to an infinite reduction of $\mathsf{mark}(t)$ in the transformed TRS, Θ_1 is *sound* for termination: Termination of the transformed TRS implies termination of the original CSRS.

In Θ_2, active can be shifted downwards to any active position. Here, the root of a term is marked with the symbol top and the symbol proper is used to check that terms only contain function symbols from the original signature.

Definition 3 (Θ_2). *Let $(\mathcal{R}, \mu)$ be a CSRS over a signature $\mathcal{F}$. The TRS $\mathcal{R}_\mu^2$ over the signature $\mathcal{F}_2 = \mathcal{F} \cup \{\mathsf{active}, \mathsf{mark}, \mathsf{top}, \mathsf{proper}, \mathsf{ok}\}$ consists of the following rules (for all $l \to r \in \mathcal{R}$, $f \in \mathcal{F}$ of arity $n > 0$, $i \in \mu(f)$, and constants $c \in \mathcal{F}$):*

$$\begin{aligned}
\mathsf{active}(l) &\to \mathsf{mark}(r) \\
\mathsf{active}(f(x_1, \dots, x_i, \dots, x_n)) &\to f(x_1, \dots, \mathsf{active}(x_i), \dots, x_n) \\
f(x_1, \dots, \mathsf{mark}(x_i), \dots, x_n) &\to \mathsf{mark}(f(x_1, \dots, x_i, \dots, x_n)) \\
\mathsf{proper}(c) &\to \mathsf{ok}(c) \\
\mathsf{proper}(f(x_1, \dots, x_n)) &\to f(\mathsf{proper}(x_1), \dots, \mathsf{proper}(x_n)) \\
f(\mathsf{ok}(x_1), \dots, \mathsf{ok}(x_n)) &\to \mathsf{ok}(f(x_1, \dots, x_n)) \\
\mathsf{top}(\mathsf{mark}(x)) &\to \mathsf{top}(\mathsf{proper}(x)) \\
\mathsf{top}(\mathsf{ok}(x)) &\to \mathsf{top}(\mathsf{active}(x))
\end{aligned}$$

The transformation $(\mathcal{R}, \mu) \mapsto \mathcal{R}_\mu^2$ is denoted by Θ_2 and we shorten $\to_{\mathcal{R}_\mu^2}$ to $\to_2$.

Transformation Θ_2 as well as the transformations[3] Θ_{L} of Lucas [23], Θ_{Z} of Zantema [32], and Θ_{FR} of Ferreira & Ribeiro [10] are sound for termination. However, only Θ_2 is *complete*, i.e., the other four transformations do not transform every terminating CSRS into a terminating TRS. The following example demonstrates the reason for the incompleteness of Θ_1.

Example 4 ([14]). Consider the non-terminating TRS $\mathcal{R}$ consisting of the rules

$$\mathsf{f}(\mathsf{b}, \mathsf{c}, x) \to \mathsf{f}(x, x, x) \qquad\qquad \mathsf{d} \to \mathsf{b} \qquad\qquad \mathsf{d} \to \mathsf{c}$$

If $\mu(\mathsf{f}) = \{3\}$ then the CSRS is terminating because the cyclic reduction of $\mathsf{f}(\mathsf{b}, \mathsf{c}, \mathsf{d})$ to $\mathsf{f}(\mathsf{d}, \mathsf{d}, \mathsf{d})$ and further to $\mathsf{f}(\mathsf{b}, \mathsf{c}, \mathsf{d})$ cannot be done, as one would have to reduce the first and second argument of f. However, the transformed TRS $\mathcal{R}_\mu^1$

$$\begin{aligned}
\mathsf{active}(\mathsf{f}(\mathsf{b}, \mathsf{c}, x)) &\to \mathsf{mark}(\mathsf{f}(x, x, x)) & \mathsf{mark}(\mathsf{f}(x, y, z)) &\to \mathsf{active}(\mathsf{f}(x, y, \mathsf{mark}(z))) \\
\mathsf{active}(\mathsf{d}) &\to \mathsf{mark}(\mathsf{b}) & \mathsf{mark}(\mathsf{b}) &\to \mathsf{active}(\mathsf{b}) \\
\mathsf{active}(\mathsf{d}) &\to \mathsf{mark}(\mathsf{c}) & \mathsf{mark}(\mathsf{c}) &\to \mathsf{active}(\mathsf{c}) \\
\mathsf{active}(x) &\to x & \mathsf{mark}(\mathsf{d}) &\to \mathsf{active}(\mathsf{d})
\end{aligned}$$

is not terminating:

$$\begin{aligned}
&\mathsf{mark}(\mathsf{f}(\mathsf{b}, \mathsf{c}, \mathsf{d})) \to_1 \mathsf{active}(\mathsf{f}(\mathsf{b}, \mathsf{c}, \mathsf{mark}(\mathsf{d}))) \to_1 \mathsf{active}(\mathsf{f}(\mathsf{b}, \mathsf{c}, \mathsf{active}(\mathsf{d}))) \\
&\to_1 \mathsf{mark}(\mathsf{f}(\mathsf{active}(\mathsf{d}), \mathsf{active}(\mathsf{d}), \mathsf{active}(\mathsf{d}))) \to_1^+ \mathsf{mark}(\mathsf{f}(\mathsf{mark}(\mathsf{b}), \mathsf{mark}(\mathsf{c}), \mathsf{d})) \\
&\to_1^+ \mathsf{mark}(\mathsf{f}(\mathsf{active}(\mathsf{b}), \mathsf{active}(\mathsf{c}), \mathsf{d})) \to_1^+ \mathsf{mark}(\mathsf{f}(\mathsf{b}, \mathsf{c}, \mathsf{d}))
\end{aligned}$$

[3] Details of the transformations Θ_{L}, Θ_{Z}, and Θ_{FR} are not needed for a proper understanding of the present paper. The interested reader is referred to [15].

Note that in the third step the 'active' subterm $\mathsf{active}(\mathsf{d})$ is copied to the first and second argument positions of f, which are inactive according to $\mu(\mathsf{f})$. This can only happen if the reduction step is non-innermost.

One should remark that transformation Θ_2 does not render the other transformations superfluous, since in practical examples, termination of $\Theta_2(\mathcal{R}, \mu)$ can be harder to show than termination of the TRSs resulting from the other transformations. In Figure 1 we compare the power of the five transformations for proving termination. Here, "*Transformation 1* $\to$ *Transformation 2*" means that *Transformation 2* is more powerful than *Transformation 1*, i.e., if Transformation 1 yields a terminating TRS, then so does Transformation 2, but not vice versa. The proofs of the various implications can be found in [15].

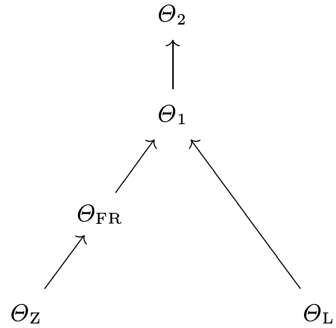

Fig. 1. Comparison of existing transformations for proving termination.

3 Innermost Termination of Context-Sensitive Rewriting

Now we examine the usefulness of the five transformations for *innermost* termination of CSRSs. Lucas [25] showed that Θ_{L} and Θ_{Z} are unsound[4] for innermost termination, i.e., innermost termination of the transformed TRS does not imply innermost termination of the original CSRS. The example showing the latter ([25, Example 12]) also demonstrates that Θ_{FR} is unsound for innermost termination. Moreover, none of these transformations is complete for innermost termination. The following new result shows that Θ_1 is sound for innermost termination.[5]

Theorem 5. *Let $(\mathcal{R}, \mu)$ be a CSRS. If $\mathcal{R}^1_\mu$ is innermost terminating then $(\mathcal{R}, \mu)$ is innermost terminating.*

[4] Θ_{L} is sound for the subclass of left-linear CSRSs with the property that all function symbols in the left-hand sides are on active positions [25].

[5] The same claim is made in [25, Theorem 11]. However, Lucas only proved the soundness of Θ_1 and Θ_2 for *ground* innermost termination (cf. Section 6) and later claimed that Θ_1 and Θ_2 are unsound for innermost termination [26].

Proof. Let $\mathcal{F}$ be the signature of $\mathcal{R}$ and let c be an arbitrary constant in $\mathcal{F}$. We show that every innermost reduction step $s \xrightarrow{\mathsf{i}}_\mu t$ in $(\mathcal{R}, \mu)$ corresponds to an innermost reduction sequence $\mathsf{mark}(s\theta){\downarrow_\mathcal{M}} \xrightarrow{\mathsf{i}}{}_1^+ \mathsf{mark}(t\theta){\downarrow_\mathcal{M}}$ in $\mathcal{R}^1_\mu$. Here θ is the substitution that maps all variables to c.[6] Note that since $\mathcal{M}$ is confluent and terminating, every term u has a unique $\mathcal{M}$-normal form $u{\downarrow_\mathcal{M}}$. First we show by induction on $u \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ that $\mathsf{mark}(u\theta){\downarrow_\mathcal{M}} \xrightarrow{\mathsf{i}}{}_1^* \mathsf{active}(u\theta)$. If u is a variable then $u\theta = c$ and thus $\mathsf{mark}(u\theta){\downarrow_\mathcal{M}} = \mathsf{active}(u\theta)$. If $u = f(u_1, \dots, u_n)$ then $\mathsf{mark}(u\theta){\downarrow_\mathcal{M}} = \mathsf{active}(f(u'_1, \dots, u'_n))$ with $u'_i = \mathsf{mark}(u_i\theta){\downarrow_\mathcal{M}}$ if $i \in \mu(f)$ and $u'_i = u_i\theta$ if $i \notin \mu(f)$. Let $i \in \mu(f)$. The induction hypothesis yields $u'_i \xrightarrow{\mathsf{i}}{}_1^* \mathsf{active}(u_i\theta)$. Since $u_i\theta$ is an $\mathcal{R}^1_\mu$-normal form, $\mathsf{active}(u_i\theta) \xrightarrow{\mathsf{i}}_1 u_i\theta$ and thus $u'_i \xrightarrow{\mathsf{i}}{}_1^* u_i\theta$. It follows that $\mathsf{mark}(u\theta){\downarrow_\mathcal{M}} \xrightarrow{\mathsf{i}}{}_1^* \mathsf{active}(f(u_1\theta, \dots, u_n\theta)) = \mathsf{active}(u\theta)$.

Now let π be the position of the redex contracted in the reduction step $s \xrightarrow{\mathsf{i}}_\mu t$. We prove the lemma by induction on π. If $\pi = \epsilon$ then $s \to t$ and thus also $s\theta \to t\theta$ is an instance of a rule in $\mathcal{R}$. We have $\mathsf{mark}(s\theta){\downarrow_\mathcal{M}} \xrightarrow{\mathsf{i}}{}_1^* \mathsf{active}(s\theta)$ by the above observation. Moreover, $\mathsf{active}(s\theta) \xrightarrow{\mathsf{i}}_1 \mathsf{mark}(t\theta)$ since $\mathsf{active}(s\theta) \to \mathsf{mark}(t\theta)$ is an instance of a rule in $\mathcal{R}^1_\mu$. We also have $\mathsf{mark}(t\theta) \xrightarrow{\mathsf{i}}{}_1^* \mathsf{mark}(t\theta){\downarrow_\mathcal{M}}$. Combining all reductions yields $\mathsf{mark}(s\theta){\downarrow_\mathcal{M}} \xrightarrow{\mathsf{i}}{}_1^+ \mathsf{mark}(t\theta){\downarrow_\mathcal{M}}$.

If $\pi = i\pi'$ then $s = f(s_1, \dots, s_i, \dots, s_n)$ and $t = f(s_1, \dots, t_i, \dots, s_n)$ with $s_i \xrightarrow{\mathsf{i}}_\mu t_i$. Note that we have $i \in \mu(f)$ due to the definition of context-sensitive rewriting. For $1 \leqslant j \leqslant n$ define $s'_j = \mathsf{mark}(s_j\theta){\downarrow_\mathcal{M}}$ if $j \in \mu(f)$ and $s'_j = s_j\theta$ if $j \notin \mu(f)$. The induction hypothesis yields $s'_i \xrightarrow{\mathsf{i}}{}_1^+ \mathsf{mark}(t_i\theta){\downarrow_\mathcal{M}}$. The result follows since $\mathsf{mark}(s\theta){\downarrow_\mathcal{M}} = \mathsf{active}(f(s'_1, \dots, s'_i, \dots, s'_n))$ and $\mathsf{mark}(t\theta){\downarrow_\mathcal{M}} = \mathsf{active}(f(s'_1, \dots, \mathsf{mark}(t_i\theta){\downarrow_\mathcal{M}}, \dots, s'_n))$. ☐

Not surprisingly, Θ_1 is incomplete for innermost termination.

Example 6 ([25]). Consider the CSRS $(\mathcal{R}, \mu)$ with $\mathcal{R}$ consisting of the rules

$$\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{a}) \qquad\qquad \mathsf{a} \to \mathsf{b}$$

and $\mu(\mathsf{f}) = \{1\}$. The CSRS $(\mathcal{R}, \mu)$ is innermost terminating but $\mathcal{R}^1_\mu$

$$\begin{aligned} \mathsf{active}(\mathsf{f}(\mathsf{a})) &\to \mathsf{mark}(\mathsf{f}(\mathsf{a})) & \mathsf{mark}(\mathsf{f}(x)) &\to \mathsf{active}(\mathsf{f}(\mathsf{mark}(x))) \\ \mathsf{active}(\mathsf{a}) &\to \mathsf{mark}(\mathsf{b}) & \mathsf{mark}(\mathsf{a}) &\to \mathsf{active}(\mathsf{a}) \\ \mathsf{active}(x) &\to x & \mathsf{mark}(\mathsf{b}) &\to \mathsf{active}(\mathsf{b}) \end{aligned}$$

is not:

$$\begin{aligned} \mathsf{active}(\mathsf{f}(\mathsf{a})) &\xrightarrow{\mathsf{i}}_1 \mathsf{mark}(\mathsf{f}(\mathsf{a})) \xrightarrow{\mathsf{i}}_1 \mathsf{active}(\mathsf{f}(\mathsf{mark}(\mathsf{a}))) \\ &\xrightarrow{\mathsf{i}}_1 \mathsf{active}(\mathsf{f}(\mathsf{active}(\mathsf{a}))) \xrightarrow{\mathsf{i}}_1 \mathsf{active}(\mathsf{f}(\mathsf{a})) \end{aligned}$$

Observe that applying the rule $\mathsf{active}(\mathsf{a}) \to \mathsf{mark}(\mathsf{b})$ instead of $\mathsf{active}(x) \to x$ in the fourth step would break the cycle. So the rule $\mathsf{active}(x) \to x$ can delete innermost redexes, causing non-innermost active redexes of the underlying CSRS to become innermost. We come back to this in Section 5.

[6] It is interesting to note that the instantiated context-sensitive reduction step $s\theta \to_\mu t\theta$ need not be innermost.

Transformation Θ_2 is sound for innermost termination as well. However, it is also incomplete and (in contrast to Θ_1) rather useless for innermost termination. These observations are consequences of the following new result. In particular, Θ_2 cannot prove innermost termination of non-terminating CSRSs.

Theorem 7. *Let $(\mathcal{R}, \mu)$ be a CSRS. The TRS $\mathcal{R}^2_\mu$ is innermost terminating if and only if it is terminating.*

Proof. Let $\mathcal{F}$ be the signature of $\mathcal{R}$. The "if" direction is trivial. For the "only if" direction suppose $\mathcal{R}^2_\mu$ is non-terminating. Since Θ_2 is complete for termination, $(\mathcal{R}, \mu)$ is non-terminating. So there exists an infinite reduction $t_1 \to_\mu t_2 \to_\mu \cdots$ consisting of ground terms from $\mathcal{T}(\mathcal{F})$. The soundness proof in [14, Theorem 3] and [15, Theorem 27] transforms this infinite reduction into the following infinite reduction in $\mathcal{R}^2_\mu$: $\mathsf{top}(\mathsf{active}(t_1)) \to_2^+ \mathsf{top}(\mathsf{active}(t_2)) \to_2^+ \cdots$. It is easy to prove that this latter reduction is actually innermost. Hence $\mathcal{R}^2_\mu$ is not innermost terminating. ☐

The soundness of Θ_2 for innermost termination is an immediate consequence of Theorem 7 and the soundness of Θ_2 for termination.

So Θ_1 is the only sound and useful transformation for innermost termination of CSRSs so far. In the remainder of this section we show that it is complete for an important subclass of CSRSs. More precisely, while in general termination of a CSRS $(\mathcal{R}, \mu)$ does not imply termination of the transformed TRS $\mathcal{R}^1_\mu$ (as demonstrated by Example 4), we show that it at least implies *innermost* termination of $\mathcal{R}^1_\mu$. This implies that for subclasses of CSRSs where innermost termination is equivalent to termination, Θ_1 is complete for innermost termination. In Section 4 we show that this subclass contains all orthogonal systems (e.g., CSRSs like Example 1 from the introduction).

We first show the desired result on innermost termination of $\mathcal{R}^1_\mu$ for those terms containing the new symbols active and mark on active positions only, except that subterms of the form $\mathsf{mark}^n(x)$ with $n \geqslant 1$ and x a variable may occur at inactive positions as well.

Lemma 8. *Let $(\mathcal{R}, \mu)$ be a terminating CSRS over a signature $\mathcal{F}$. Let $t \in \mathcal{T}(\mathcal{F}_1, \mathcal{V})$ where* active *and* mark *occur on active positions in t only (here the argument positions of* active *and* mark *are also considered active), except that t may contain subterms of the form* $\mathsf{mark}^n(x)$ *with $x \in \mathcal{V}$ at inactive positions. Then t is $\mathcal{R}^1_\mu$-terminating.*

Proof. Let $\mathcal{M}_1 = \mathcal{M} \cup \{\mathsf{active}(x) \to x\}$. Note that $\mathcal{M}_1$ is confluent and terminating. Hence, every infinite $\mathcal{R}^1_\mu$-reduction contains infinitely many reduction steps with rules from $\mathcal{R}^1_\mu \setminus \mathcal{M}_1$. Let T_1 be the set of all terms t described above. It is not difficult to see that $t \to_1 u$ and $t \in T_1$ imply $u \in T_1$. Let $\mathcal{M}'$ be the confluent and terminating TRS consisting of the rules $\mathsf{active}(x) \to x$ and $\mathsf{mark}(x) \to x$. Clearly, $t \to_{\mathcal{M}_1} u$ implies $t{\downarrow}_{\mathcal{M}'} = u{\downarrow}_{\mathcal{M}'}$. We show that for all

$t \in T_1$, $t \to_{\mathcal{R}_\mu^1 \setminus \mathcal{M}_1} u$ implies $t{\downarrow_{\mathcal{M}'}} \to_\mu u{\downarrow_{\mathcal{M}'}}$. Since $\mathcal{M}_1$ is terminating, every infinite $\mathcal{R}_\mu^1$-reduction starting from T_1 can be transformed into an infinite reduction in $(\mathcal{R}, \mu)$, which proves the lemma. From $t \to_{\mathcal{R}_\mu^1 \setminus \mathcal{M}_1} u$ we infer the existence of a position π in t, a rewrite rule $l \to r \in \mathcal{R}$, and a substitution σ such that $t|_\pi = \mathsf{active}(l\sigma)$ and $u = t[\mathsf{mark}(r\sigma)]_\pi$. Since $t \in T_1$, π is an active position in t. We have $t{\downarrow_{\mathcal{M}'}} = t{\downarrow_{\mathcal{M}'}}[l\sigma']_{\pi'}$ and $u{\downarrow_{\mathcal{M}'}} = t{\downarrow_{\mathcal{M}'}}[r\sigma']_{\pi'}$ for some active position π' and the substitution σ' with $\sigma'(x) = \sigma(x){\downarrow_{\mathcal{M}'}}$. Therefore, $t{\downarrow_{\mathcal{M}'}} \to_\mu u{\downarrow_{\mathcal{M}'}}$. □

Now we can show that for a terminating CSRS, the transformed TRS is at least innermost terminating.

Theorem 9. *Let $(\mathcal{R}, \mu)$ be a CSRS. If $(\mathcal{R}, \mu)$ is terminating then $\mathcal{R}_\mu^1$ is innermost terminating.*

Proof. Let $\mathcal{F}$ be the signature of $\mathcal{R}$. Let $\#(t)$ denote the number of active and mark-symbols occurring in the term $t \in \mathcal{T}(\mathcal{F}_1, \mathcal{V})$, except that we do not count the occurrences of mark in subterms of the form $\mathsf{mark}^n(x)$. We prove that t is innermost $\mathcal{R}_\mu^1$-terminating by induction on $\#(t)$. If $\#(t) = 0$ then t is an $\mathcal{R}_\mu^1$-normal form. If $\#(t) > 0$ then t must contain an innermost $\mathcal{R}_\mu^1$-redex, say at position π. We have $t|_\pi = \mathsf{active}(t')$ or $t|_\pi = \mathsf{mark}(t')$ such that t' does not contain any active-symbols and the only mark-symbols occurring in t' are in subterms of the form $\mathsf{mark}^n(x)$ (hence, $\#(t|_\pi) = 1$). It follows that Lemma 8 is applicable to $t|_\pi$. So $t|_\pi$ does not admit infinite $\mathcal{R}_\mu^1$-reductions. To conclude that t is innermost $\mathcal{R}_\mu^1$-terminating, it suffices to show that $t[u]_\pi$ is innermost $\mathcal{R}_\mu^1$-terminating for every normal form u of $t|_\pi$ reachable by innermost $\mathcal{R}_\mu^1$-reductions. Since $\#(u) = 0$, $\#(t) > \#(t[u]_\pi)$ and thus the result follows from the induction hypothesis. □

So for a terminating CSRS $(\mathcal{R}, \mu)$, non-termination of $\mathcal{R}_\mu^1$ can only be due to the rewriting strategy. This provides further evidence for the power of Θ_1. Note that this result does not hold for the transformations of Lucas, Zantema, and Ferreira & Ribeiro. The CSRS $(\mathcal{R}, \mu)$ with the rules $\mathcal{R} = \{\mathsf{g}(x) \to \mathsf{h}(x), \mathsf{c} \to \mathsf{d}, \mathsf{h}(\mathsf{d}) \to \mathsf{g}(\mathsf{c})\}$ and $\mu(\mathsf{g}) = \mu(\mathsf{h}) = \varnothing$ from [32] is terminating, but none of the TRSs $\Theta_{\mathrm{L}}(\mathcal{R}, \mu)$, $\Theta_{\mathrm{Z}}(\mathcal{R}, \mu)$, and $\Theta_{\mathrm{FR}}(\mathcal{R}, \mu)$ is even innermost terminating. On the other hand, $\Theta_1(\mathcal{R}, \mu) = \mathcal{R}_\mu^1$ is (innermost) terminating [14].

4 Termination versus Innermost Termination

There are two motivations for studying innermost termination of CSRSs. First, innermost context-sensitive rewriting models evaluation in OBJ and related languages and thus, techniques for innermost termination analysis of CSRSs can be used for termination analysis of OBJ-programs. But second, techniques for innermost termination analysis of CSRSs can also be helpful for (non-innermost) termination proofs of CSRSs. This is similar to the situation with ordinary term rewriting: Proving innermost termination is much easier than proving termination, cf. [1]. There are classes of TRSs where innermost termination already implies termination and therefore for such systems, one should rather use innermost termination techniques for investigating their termination behavior.

In order to use a corresponding approach for context-sensitive rewriting, in this section we examine the connection between termination and innermost termination for CSRSs. In general, termination implies innermost termination, but not vice versa as demonstrated by Example 6. For ordinary TRSs, Gramlich [18, Theorem 3.23] showed that termination and innermost termination coincide for the class of locally confluent overlay systems. Non-overlapping rewrite systems are locally confluent overlay systems. Hence, this provides a simple syntactic criterion to identify classes of TRSs where innermost termination suffices for termination. Unfortunately, as noted by Lucas [26], this criterion cannot be extended to context-sensitive systems.

Example 10 ([26]). Consider the CSRS $(\mathcal{R}, \mu)$ with $\mathcal{R}$ consisting of the rules

$$\mathsf{f}(x,x) \to \mathsf{b} \qquad \mathsf{f}(x,\mathsf{g}(x)) \to \mathsf{f}(x,x) \qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

and $\mu(\mathsf{f}) = \{1,2\}$, $\mu(\mathsf{g}) = \varnothing$. The CSRS $(\mathcal{R}, \mu)$ is non-overlapping and innermost terminating, but not terminating since $\mathsf{f}(\mathsf{c},\mathsf{c}) \to_\mu \mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{c})) \to_\mu \mathsf{f}(\mathsf{c},\mathsf{c}) \to_\mu \cdots$. On the other hand, in an innermost reduction we would have $\mathsf{f}(\mathsf{c},\mathsf{c}) \xrightarrow{\mathsf{i}}_\mu \mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{c})) \xrightarrow{\mathsf{i}}_\mu \mathsf{f}(\mathsf{g}(\mathsf{c}),\mathsf{g}(\mathsf{c})) \xrightarrow{\mathsf{i}}_\mu \mathsf{b}$.

So non-overlappingness is not sufficient for CSRSs in order to use innermost termination techniques for termination analysis. Below we show the new result that the desired equivalence between innermost and full termination at least holds for *orthogonal* CSRSs. Thus, this includes all CSRSs which correspond to typical functional programs like Example 1. Theorem 13 states that for such systems we only have to prove innermost termination in order to verify their termination.

In order to prove the theorem, we need some preliminaries. For non-overlapping CSRSs $(\mathcal{R}, \mu)$ the relation $\xrightarrow{\mathsf{i}}_\mu$ is confluent. Hence, for every term s there is at most one μ-normal form reachable by innermost reductions. We call this term the *innermost μ-normal form* of s and denote it by $s{\downarrow}^{\mathsf{i}}_\mu$. Now for any term s, let $\nabla(s)$ be the set of those terms which result from repeatedly replacing subterms of s by their innermost μ-normal form (if it exists). Here, one may also consider subterms on inactive positions. However, the replacement must go "from the inside to the outside" (i.e., after replacing at position π one cannot replace at positions below π any more). Moreover, one may only perform replacements on such positions π where the original term $s|_\pi$ is terminating.

Definition 11. *Let $(\mathcal{R}, \mu)$ be a non-overlapping CSRS. For any term s we define non-empty sets $\nabla(s)$ and $\nabla'(s)$ as follows. If s is terminating, then $\nabla(s) = \nabla'(s) \cup \{u{\downarrow}^{\mathsf{i}}_\mu \mid u \in \nabla'(s)$ is innermost terminating$\}$. Otherwise, we have $\nabla(s) = \nabla'(s)$. Moreover, $\nabla'(s) = \{f(u_1,\ldots,u_n) \mid u_i \in \nabla(s_i)\}$ if $s = f(s_1,\ldots,s_n)$ and $\nabla'(s) = \{s\}$ if s is a variable.*

The following auxiliary lemma describes how ∇ operates on instantiated subterms of left-hand sides.

Lemma 12. *Let $(\mathcal{R}, \mu)$ be an orthogonal CSRS, let t be a proper subterm of a left-hand side of a rule, and let $u \in \nabla(t\sigma)$ for a substitution σ. Then we have $u = t\sigma'$ for some substitution σ'. Moreover, for all $x \in \mathcal{V}\mathrm{ar}(t)$ we have $x\sigma' \in \nabla(x\sigma)$ and if $u \in \nabla'(t\sigma)$ then we also have $x\sigma' \in \nabla'(x\sigma)$.*

Proof. The lemma is proved by structural induction on t. If $t = x \in \mathcal{V}$ then the claim is obvious for the substitution σ' that replaces x by u. Now let $t = f(t_1, \ldots, t_n)$. We first regard the case where $u \in \nabla'(t\sigma)$. So $u = f(u_1, \ldots, u_n)$ and $u_i \in \nabla(t_i\sigma)$ for all i. The induction hypothesis states that $u_i = t_i\sigma'$ for all i. Note that we can use the same substitution σ' for every i since t is linear due to the orthogonality of $(\mathcal{R}, \mu)$. The induction hypothesis also implies that we have $x\sigma' \in \nabla'(x\sigma)$ for all $x \in \mathcal{V}\mathrm{ar}(t_1) \cup \cdots \cup \mathcal{V}\mathrm{ar}(t_n) = \mathcal{V}\mathrm{ar}(t)$. In the remaining case $t\sigma$ is terminating and $u = v{\downarrow}^{\mathsf{i}}_{\mu}$ for some $v \in \nabla'(t\sigma)$ which is innermost terminating. Similar as in the previous case, the induction hypothesis states that $v = t\sigma'$ for some substitution σ' and $x\sigma' \in \nabla'(x\sigma)$ for all $x \in \mathcal{V}\mathrm{ar}(t)$. We define the substitution σ'' as

$$\sigma''(x) = \begin{cases} x\sigma'{\downarrow}^{\mathsf{i}}_{\mu} & \text{if } x \text{ is at an active position in } t \\ x\sigma' & \text{otherwise} \end{cases}$$

The substitution σ'' is well defined, because if x occurs at an active position in t, then $x\sigma'$ occurs at an active position in $t\sigma' = v$ and hence, innermost termination of v implies innermost termination of $x\sigma'$. Since non-variable subterms at active positions in t do not unify with left-hand sides due to the orthogonality of $(\mathcal{R}, \mu)$, we have $u = v{\downarrow}^{\mathsf{i}}_{\mu} = t\sigma'{\downarrow}^{\mathsf{i}}_{\mu} = t\sigma''$. Let $x \in \mathcal{V}\mathrm{ar}(t)$. If x occurs at an active position in t then termination of $x\sigma$ follows from termination of $t\sigma$. Thus, $x\sigma'' = x\sigma'{\downarrow}^{\mathsf{i}}_{\mu} \in \nabla(x\sigma)$ since $x\sigma' \in \nabla'(x\sigma)$. If x occurs only at inactive positions in t then $x\sigma' \in \nabla'(x\sigma)$ trivially implies $x\sigma'' = x\sigma' \in \nabla'(x\sigma) \subseteq \nabla(x\sigma)$. Thus, σ'' is a substitution as required in the lemma. $\square$

Now we show the desired theorem on the equivalence of innermost and full termination.

Theorem 13. *An orthogonal CSRS $(\mathcal{R}, \mu)$ is terminating if and only if it is* innermost *terminating.*

Proof. The "only if" direction is trivial. We prove the "if" direction. Let $s \to_\mu t$ where the contracted redex is either terminating or a minimal non-terminating term (i.e., all proper subterms of the redex on active positions are terminating). We prove the following statements for all innermost terminating $s' \in \nabla(s)$:

(1) There exists a $t' \in \nabla(t)$ such that $s' \xrightarrow{\mathsf{i}}{}^{*}_{\mu} t'$.
(2) If the contracted redex in $s \to_\mu t$ is not terminating, then there even exists a $t' \in \nabla(t)$ such that $s' \xrightarrow{\mathsf{i}}{}^{+}_{\mu} t'$.

With (1) and (2) one can prove the theorem: If $(\mathcal{R}, \mu)$ is not terminating, then there is an infinite reduction $s_0 \to_\mu s_1 \to_\mu \ldots$ in which only terminating or minimal non-terminating redexes are contracted. Assume that $(\mathcal{R}, \mu)$ is innermost terminating. Then all $\nabla(s_i)$ contain only innermost terminating terms and since $s_0 \in \nabla(s_0)$, we can construct an infinite innermost reduction $s_0 \xrightarrow{\mathsf{i}}{}_\mu^* t_1 \xrightarrow{\mathsf{i}}{}_\mu^* t_2 \xrightarrow{\mathsf{i}}{}_\mu^* \ldots$ with $t_i \in \nabla(s_i)$. However, since the reduction contains infinitely many steps of type (2), this gives rise to an infinite innermost reduction, contradicting our assumption.

Now we prove (1) and (2) by structural induction on s. Since $s \to_\mu t$, s must have the form $f(s_1, \ldots, s_n)$. We first regard the case where $s \to_\mu t$ is not a root reduction step. Then we have $t = f(s_1, \ldots, t_i, \ldots, s_n)$ with $s_i \to_\mu t_i$ for some $i \in \mu(f)$. Let $s' \in \nabla(s)$ be innermost terminating. First, let $s' = f(u_1, \ldots, u_n)$ with $u_j \in \nabla(s_j)$ for all j. Because $i \in \mu(f)$, u_i is innermost terminating. Hence by the induction hypothesis, $u_i \in \nabla(s_i)$ implies that there exists a $v_i \in \nabla(t_i)$ such that $u_i \xrightarrow{\mathsf{i}}{}_\mu^* v_i$. Therefore, we also have $s' = f(u_1, \ldots, u_i, \ldots, u_n) \xrightarrow{\mathsf{i}}{}_\mu^* f(u_1, \ldots, v_i, \ldots, u_n) \in \nabla(t)$. Moreover, if the contracted redex in $s \to_\mu t$ and hence, in $s_i \to_\mu t_i$ is not terminating, then by the induction hypothesis we even have $u_i \xrightarrow{\mathsf{i}}{}_\mu^+ v_i$ and therefore $s' \xrightarrow{\mathsf{i}}{}_\mu^+ f(u_1, \ldots, v_i, \ldots, u_n) \in \nabla(t)$.

Now let $s' = f(u_1, \ldots, u_n){\downarrow}^{\mathsf{i}}_\mu$ with $u_j \in \nabla(s_j)$ for all j. Hence, s is terminating and thus, we only have to prove (1). As before, there is a $v_i \in \nabla(t_i)$ such that $u_i \xrightarrow{\mathsf{i}}{}_\mu^* v_i$ and $f(u_1, \ldots, v_i, \ldots, u_n) \in \nabla(t)$. Since innermost reduction is confluent, we have $s' = f(u_1, \ldots, u_i, \ldots, u_n){\downarrow}^{\mathsf{i}}_\mu = f(u_1, \ldots, v_i, \ldots, u_n){\downarrow}^{\mathsf{i}}_\mu \in \nabla(t)$, since t inherits termination from s.

Finally, we regard the case where $s = f(s_1, \ldots, s_n)$ and $s \to_\mu t$ is a root reduction step. Hence, there must be a rule $l \to r \in \mathcal{R}$ with $l = f(l_1, \ldots, l_n)$ and a substitution σ such that $s_i = l_i\sigma$ and $t = r\sigma$. First let $s' = f(u_1, \ldots, u_n)$ with $u_i \in \nabla(s_i)$ for all i. Since $(\mathcal{R}, \mu)$ is orthogonal and since $s_i = l_i\sigma$, due to Lemma 12 there must be a substitution σ' such that $u_i = l_i\sigma'$ for all i. Because s' is innermost terminating, $x\sigma'$ must also be innermost terminating for all variables x which occur on active positions of l. Let σ'' be the substitution where $x\sigma'' = x\sigma'{\downarrow}^{\mathsf{i}}_\mu$ for all x in active positions of l and $x\sigma'' = x\sigma'$ for all other x. Then we have the innermost reduction $s' = f(l_1\sigma', \ldots, l_n\sigma') \xrightarrow{\mathsf{i}}{}_\mu^* f(l_1\sigma'', \ldots, l_n\sigma'') \xrightarrow{\mathsf{i}}{}_\mu r\sigma''$. We claim that $r\sigma'' \in \nabla(t) = \nabla(r\sigma)$. To this end, it suffices to show that $x\sigma'' \in \nabla(x\sigma)$ for all variables x in r, because in the construction of ∇ arbitrary subterms q can be replaced by terms from $\nabla(q)$. Each variable x occurs in some l_i and we have $l_i\sigma' \in \nabla(l_i\sigma)$. From Lemma 12 we obtain $x\sigma' \in \nabla(x\sigma)$ for all variables x. If x is on an inactive position of l, then $x\sigma'' = x\sigma' \in \nabla(x\sigma)$. If x is on an active position of l, then $x\sigma'' = x\sigma'{\downarrow}^{\mathsf{i}}_\mu \in \nabla(x\sigma)$, since $x\sigma'$ is innermost terminating and because in this case, $x\sigma$ is terminating due to the fact that s is either a terminating or a *minimal* non-terminating term.

Now let $s' = f(u_1, \ldots, u_n){\downarrow}^{\mathsf{i}}_\mu$ with $u_i \in \nabla(s_i)$ for all i. Hence, s is terminating and thus we only have to prove (1). As before, $u_i = l_i\sigma'$ and $f(l_1\sigma', \ldots, l_n\sigma') \xrightarrow{\mathsf{i}}{}_\mu^* f(l_1\sigma'', \ldots, l_n\sigma'') \xrightarrow{\mathsf{i}}{}_\mu r\sigma''$ with $r\sigma'' \in \nabla(t)$. Since innermost reduction is confluent and t inherits termination from s, $s' = f(u_1, \ldots, u_n){\downarrow}^{\mathsf{i}}_\mu = r\sigma''{\downarrow}^{\mathsf{i}}_\mu \in \nabla(t)$. $\square$

Very recently, Gramlich and Lucas [19] showed that termination and innermost termination coincide for locally confluent overlay CSRSs with the additionally property that variables that occur at an active position in a left-hand side l of a rewrite rule $l \to r$ do not occur at inactive positions in l or r. The latter condition is quite restrictive, e.g., it is not satisfied by the CSRS of Example 1, since in the rule for "$-$" the variables x and y occur on active positions in the left-hand side, but also on inactive positions in the right-hand side.

5 A Sound and Complete Transformation

In Section 3 we have seen that none of the existing transformations is complete for innermost termination and that only Θ_1 and Θ_2 are sound. Because of Theorem 7, Θ_2 cannot distinguish innermost termination from termination. So when trying to develop a sound and complete transformation for innermost termination, we take Θ_1 as starting point. As observed in Example 6, we must make sure that in innermost reductions, rules of the form $\mathsf{active}(l) \to \mathsf{mark}(r)$ get preference over the rule $\mathsf{active}(x) \to x$, because then this counterexample no longer works. Hence, we modify the rule $\mathsf{active}(x) \to x$ in such a way that the innermost reduction strategy ensures that $\mathsf{active}(l) \to \mathsf{mark}(r)$ is applied with higher preference. In the modification, $\mathsf{active}(l) \to \mathsf{mark}(r)$ no longer overlaps with the *root position* of $\mathsf{active}(x) \to x$, but with a non-root position of the new modified rule(s).

Definition 14 (Θ_3). *Let $(\mathcal{R}, \mu)$ be a CSRS over a signature $\mathcal{F}$. The TRS $\mathcal{R}^3_\mu$ over the signature $\mathcal{F}_1 = \mathcal{F} \cup \{\mathsf{active}, \mathsf{mark}\}$ consists of the following rewrite rules (for all $l \to r \in \mathcal{R}$, $f \in \mathcal{F}$, and $1 \leqslant i \leqslant \mathrm{arity}(f)$):*

$$\begin{aligned} \mathsf{active}(l) &\to \mathsf{mark}(r) \\ \mathsf{mark}(f(x_1, \ldots, x_n)) &\to \mathsf{active}(f([x_1]^f_1, \ldots, [x_n]^f_n)) \\ f(x_1, \ldots, \mathsf{active}(x_i), \ldots, x_n) &\to f(x_1, \ldots, x_i, \ldots, x_n) \\ f(x_1, \ldots, \mathsf{mark}(x_i), \ldots, x_n) &\to f(x_1, \ldots, x_i, \ldots, x_n) \qquad (\flat) \end{aligned}$$

Again, $[t]^f_i = \mathsf{mark}(t)$ if $i \in \mu(f)$ and $[t]^f_i = t$ otherwise. We denote the transformation $(\mathcal{R}, \mu) \mapsto \mathcal{R}^3_\mu$ by Θ_3 and we abbreviate $\to_{\mathcal{R}^3_\mu}$ to $\to_3$ and $\xrightarrow{\mathsf{i}}_{\mathcal{R}^3_\mu}$ to $\xrightarrow{\mathsf{i}}_3$.

For the CSRS $(\mathcal{R}, \mu)$ of Example 6, $\mathcal{R}^3_\mu$ differs from $\mathcal{R}^1_\mu$ in two respects: $\mathsf{active}(x) \to x$ is replaced by $\mathsf{f}(\mathsf{active}(x)) \to \mathsf{f}(x)$ and moreover, the rule $\mathsf{f}(\mathsf{mark}(x)) \to \mathsf{f}(x)$ is added. As a consequence, the cycle $\mathsf{active}(\mathsf{f}(\mathsf{a})) \xrightarrow{\mathsf{i}}^+ \mathsf{active}(\mathsf{f}(\mathsf{a}))$ can no longer be obtained with $\mathcal{R}^3_\mu$, since $\mathsf{active}(\mathsf{f}(\mathsf{active}(\mathsf{a}))) \to \mathsf{active}(\mathsf{f}(\mathsf{a}))$ is not an *innermost* rewrite step in $\mathcal{R}^3_\mu$. Indeed, $\mathcal{R}^3_\mu$ is innermost terminating and in general, Θ_3 is sound and complete for innermost termination.

With the new rules $f(x_1, \ldots, \mathsf{active}(x_i), \ldots, x_n) \to f(x_1, \ldots, x_n)$ we can remove almost every active-symbol, compensating to a large extent the lack of the rule $\mathsf{active}(x) \to x$. The ($\flat$)-marked rules can never be used in an innermost reduction if x_i is instantiated to a non-variable term from $\mathcal{T}(\mathcal{F}, \mathcal{V})$. However, they are required if x_i is instantiated by a variable or by terms containing the symbols mark and active. As a matter of fact, the transformation without these rules is neither sound nor complete for innermost termination.

Example 15. Consider the CSRS $(\mathcal{R}, \mu)$ with $\mathcal{R}$ consisting of the four rules

$$\begin{aligned} \mathsf{g}(\mathsf{f}(x,x)) &\to \mathsf{g}(\mathsf{f}(x,x)) & \qquad \mathsf{f}(\mathsf{b},x) &\to \mathsf{b} \\ \mathsf{f}(\mathsf{g}(x),y) &\to \mathsf{b} & \mathsf{f}(\mathsf{f}(x,y),z) &\to \mathsf{b} \end{aligned}$$

and $\mu(\mathsf{f}) = \mu(\mathsf{g}) = \{1\}$. The CSRS $(\mathcal{R}, \mu)$ is not innermost terminating as $\mathsf{g}(\mathsf{f}(x,x)) \xrightarrow{\mathsf{i}}_\mu \mathsf{g}(\mathsf{f}(x,x))$. The transformed TRS $\mathcal{R}^3_\mu$

$$\begin{aligned} \mathsf{active}(\mathsf{g}(\mathsf{f}(x,x))) &\to \mathsf{mark}(\mathsf{g}(\mathsf{f}(x,x))) & \qquad \mathsf{mark}(\mathsf{b}) &\to \mathsf{active}(\mathsf{b}) \\ \mathsf{active}(\mathsf{f}(\mathsf{b},x)) &\to \mathsf{mark}(\mathsf{b}) & \mathsf{mark}(\mathsf{f}(x,y)) &\to \mathsf{active}(\mathsf{f}(\mathsf{mark}(x),y)) \\ \mathsf{active}(\mathsf{f}(\mathsf{g}(x),y)) &\to \mathsf{mark}(\mathsf{b}) & \mathsf{mark}(\mathsf{g}(x)) &\to \mathsf{active}(\mathsf{g}(\mathsf{mark}(x))) \\ \mathsf{active}(\mathsf{f}(\mathsf{f}(x,y),z)) &\to \mathsf{mark}(\mathsf{b}) & & \\ \mathsf{f}(\mathsf{active}(x),y) &\to \mathsf{f}(x,y) & \mathsf{f}(\mathsf{mark}(x),y) &\to \mathsf{f}(x,y) \quad (*) \\ \mathsf{f}(x,\mathsf{active}(y)) &\to \mathsf{f}(x,y) & \mathsf{f}(x,\mathsf{mark}(y)) &\to \mathsf{f}(x,y) \quad (*) \\ \mathsf{g}(\mathsf{active}(x)) &\to \mathsf{g}(x) & \mathsf{g}(\mathsf{mark}(x)) &\to \mathsf{g}(x) \quad\;\; (*) \end{aligned}$$

also fails to be innermost terminating:

$$\begin{aligned} \mathsf{active}(\mathsf{g}(\mathsf{f}(x,x))) &\xrightarrow{\mathsf{i}}_3 \mathsf{mark}(\mathsf{g}(\mathsf{f}(x,x))) \xrightarrow{\mathsf{i}}_3 \mathsf{active}(\mathsf{g}(\mathsf{mark}(\mathsf{f}(x,x)))) \\ &\xrightarrow{\mathsf{i}}_3 \mathsf{active}(\mathsf{g}(\mathsf{active}(\mathsf{f}(\mathsf{mark}(x),x)))) \\ &\xrightarrow{\mathsf{i}}_3 \mathsf{active}(\mathsf{g}(\mathsf{active}(\mathsf{f}(x,x)))) \xrightarrow{\mathsf{i}}_3 \mathsf{active}(\mathsf{g}(\mathsf{f}(x,x))) \end{aligned}$$

However, the TRS without the three rules marked with $(*)$ is innermost terminating. In other words, if the $(\flat)$-rules were missing, then the transformation Θ_3 would be unsound for innermost termination.

Termination of $\mathcal{R}^3_\mu$ without the $(*)$-rules can be proved as follows. By a minimality argument, it is sufficient to show that all terms t whose arguments are in normal form are innermost terminating. Let $\#(t)$ denote the number of occurrences of the function symbols b, f, and g in t. Inspection of the rewrite rules reveals that this number does not increase along a reduction. We use induction on $\#(t)$. If $\#(t) = 0$ then t is a normal form. Suppose $\#(t) > 0$. We distinguish the following five cases, depending on the root symbol of t.

1. If $t = \mathsf{b}$ then t is a normal form.
2. If $t = \mathsf{f}(t_1, t_2)$ is not in normal form then t can only be reduced by the rule $\mathsf{f}(\mathsf{active}(x), y) \to \mathsf{f}(x,y)$ or the rule $\mathsf{f}(x, \mathsf{active}(y)) \to \mathsf{f}(x,y)$. After an application of one of these rules, the arguments of the resulting term remain in normal form. It follows that any (innermost) reduction starting from t consists entirely of root reduction steps. Since the two rules decrease the size of terms, it follows that t is (innermost) terminating.

3. If $t = \mathsf{g}(t_1)$ then we obtain the innermost termination of t as in the previous case.
4. If $t = \mathsf{active}(t_1)$ is not a normal form then $t_1 = \mathsf{f}(\mathsf{b}, u)$, $t_1 = \mathsf{f}(\mathsf{g}(u_1), u_2)$, $t_1 = \mathsf{f}(\mathsf{f}(u_1, u_2), u_3)$, or $t_1 = \mathsf{g}(\mathsf{f}(u, u))$. In the first three cases there are at most two (innermost) reduction steps: $t \xrightarrow{\mathsf{i}} \mathsf{mark}(\mathsf{b}) \xrightarrow{\mathsf{i}} \mathsf{active}(\mathsf{b})$. In the fourth case, any infinite innermost reduction starting from t begins as follows:
$$\begin{aligned} t &\xrightarrow{\mathsf{i}}_1 \mathsf{mark}(\mathsf{g}(\mathsf{f}(u, u))) \\ &\xrightarrow{\mathsf{i}}_1 \mathsf{active}(\mathsf{g}(\mathsf{mark}(\mathsf{f}(u, u)))) \\ &\xrightarrow{\mathsf{i}}_1 \mathsf{active}(\mathsf{g}(\mathsf{active}(\mathsf{f}(\mathsf{mark}(u), u)))) \end{aligned}$$
If $\mathsf{mark}(u)$ is a normal form then $\mathsf{active}(\mathsf{g}(\mathsf{active}(\mathsf{f}(\mathsf{mark}(u), u))))$ reduces only to the normal form $\mathsf{active}(\mathsf{g}(\mathsf{f}(\mathsf{mark}(u), u))))$. So suppose that $\mathsf{mark}(u)$ is reducible, which implies $\mathrm{root}(u) \in \{\mathsf{b}, \mathsf{f}, \mathsf{g}\}$. We have $\#(t) > \#(\mathsf{mark}(u))$ and hence $\mathsf{mark}(u)$ is innermost terminating by the induction hypothesis. Let u' be an arbitrary normal form of $\mathsf{mark}(u)$. It suffices to show that $t' = \mathsf{active}(\mathsf{g}(\mathsf{active}(\mathsf{f}(u', u))))$ is innermost terminating. We have $u' = \mathsf{active}(\mathsf{b})$, $u' = \mathsf{active}(\mathsf{f}(v_1, v_2))$, or $u' = \mathsf{active}(\mathsf{g}(v))$. Hence, by two innermost reduction steps, we obtain $\mathsf{active}(\mathsf{g}(\mathsf{mark}(\mathsf{b})))$. Since $\#(t) > 2 = \#(\mathsf{active}(\mathsf{g}(\mathsf{mark}(\mathsf{b}))))$, the result follows from the induction hypothesis.
5. If $t = \mathsf{mark}(t_1)$ is not in normal form then by performing one (innermost) reduction step we obtain a term of the form $u = \mathsf{active}(u_1)$ with $\#(t) = \#(u)$. Hence innermost termination of t reduces to the previous case.

Example 16. Consider the CSRS $(\mathcal{R}, \mu)$ with the rules
$$\mathsf{f}(x, x) \to \mathsf{b} \qquad\qquad \mathsf{g}(\mathsf{f}(x, y)) \to \mathsf{g}(\mathsf{f}(y, y))$$
and $\mu(\mathsf{f}) = \mu(\mathsf{g}) = \{1\}$. The CSRS $(\mathcal{R}, \mu)$ is innermost terminating. The transformed TRS $\mathcal{R}^3_\mu$
$$\begin{array}{rlrl} \mathsf{active}(\mathsf{f}(x, x)) &\to \mathsf{mark}(\mathsf{b}) & \mathsf{mark}(\mathsf{b}) &\to \mathsf{active}(\mathsf{b}) \\ \mathsf{active}(\mathsf{g}(\mathsf{f}(x, y))) &\to \mathsf{mark}(\mathsf{g}(\mathsf{f}(y, y))) & \mathsf{mark}(\mathsf{f}(x, y)) &\to \mathsf{active}(\mathsf{f}(\mathsf{mark}(x), y)) \\ & & \mathsf{mark}(\mathsf{g}(x)) &\to \mathsf{active}(\mathsf{g}(\mathsf{mark}(x))) \\ \mathsf{f}(\mathsf{active}(x), y) &\to \mathsf{f}(x, y) & \mathsf{f}(\mathsf{mark}(x), y) &\to \mathsf{f}(x, y) \quad (*) \\ \mathsf{f}(x, \mathsf{active}(y)) &\to \mathsf{f}(x, y) & \mathsf{f}(x, \mathsf{mark}(y)) &\to \mathsf{f}(x, y) \quad (*) \\ \mathsf{g}(\mathsf{active}(x)) &\to \mathsf{g}(x) & \mathsf{g}(\mathsf{mark}(x)) &\to \mathsf{g}(x) \quad\;\; (*) \end{array}$$
is also innermost terminating. However, the TRS without the three rules marked with $(*)$ is *not* innermost terminating as can be seen from the following cycle, with $t = \mathsf{mark}(\mathsf{active}(\mathsf{b}))$:
$$\begin{aligned} \mathsf{mark}(\mathsf{g}(\mathsf{f}(t, t))) &\xrightarrow{\mathsf{i}}{}^{+} \mathsf{active}(\mathsf{g}(\mathsf{active}(\mathsf{f}(\mathsf{mark}(t), t)))) \\ &\xrightarrow{\mathsf{i}} \mathsf{active}(\mathsf{g}(\mathsf{f}(\mathsf{mark}(t), t))) \xrightarrow{\mathsf{i}} \mathsf{mark}(\mathsf{g}(\mathsf{f}(t, t))) \end{aligned}$$
Thus, without the $(\flat)$-marked rules, the transformation Θ_3 would be incomplete for innermost termination.

Now we prove that Θ_3 is sound and complete for innermost termination. For soundness we show that every context-sensitive innermost reduction step $s \xrightarrow{\mathsf{i}}_\mu t$ corresponds to a reduction $\mathsf{mark}(s){\downarrow_\mathcal{M}} \xrightarrow{\mathsf{i}}{}^+_3 \mathsf{mark}(t){\downarrow_\mathcal{M}}$ in the transformed system. The next lemma is used when s is an innermost μ-redex.

Lemma 17. *If $s \in \mathcal{T}(\mathcal{F},\mathcal{V}) \setminus \mathcal{V}$ such that all active arguments of s are in μ-normal form then $\mathsf{mark}(s){\downarrow_\mathcal{M}} \xrightarrow{\mathsf{i}}{}^*_3 \mathsf{active}(s)$.*

Proof. We prove the lemma by structural induction on s. Let $s = f(s_1,\ldots,s_n)$. We have $\mathsf{mark}(s){\downarrow_\mathcal{M}} = \mathsf{active}(f([s_1]^f_1{\downarrow_\mathcal{M}},\ldots,[s_n]^f_n{\downarrow_\mathcal{M}}))$. If $i \in \mu(f)$ and $s_i \notin \mathcal{V}$ then $[s_i]^f_i{\downarrow_\mathcal{M}} = \mathsf{mark}(s_i){\downarrow_\mathcal{M}} \xrightarrow{\mathsf{i}}{}^*_3 \mathsf{active}(s_i)$ according to the induction hypothesis, which is applicable since s_i is an active argument of s. Note that in this case $\mathsf{active}(s_i)$ is an $\mathcal{R}^3_\mu$-normal form because s_i is not a redex (with respect to $\mathcal{R}$). If $i \in \mu(f)$ and $s_i \in \mathcal{V}$ then $[s_i]^f_i{\downarrow_\mathcal{M}} = \mathsf{mark}(s_i)$, which is clearly an $\mathcal{R}^3_\mu$-normal form. If $i \notin \mu(f)$ then $[s_i]^f_i{\downarrow_\mathcal{M}} = s_i{\downarrow_\mathcal{M}} = s_i$. So we obtain $\mathsf{mark}(s){\downarrow_\mathcal{M}} \xrightarrow{\mathsf{i}}{}^*_3 \mathsf{active}(f(t_1,\ldots,t_n))$ where, for all $1 \leqslant i \leqslant n$, either $t_i = \mathsf{active}(s_i)$, $t_i = \mathsf{mark}(s_i)$, or $t_i = s_i$. Moreover, in the first two cases, t_i is an $\mathcal{R}^3_\mu$-normal form. Hence, by applications of the rules

$$f(x_1,\ldots,\mathsf{active}(x_i),\ldots,x_n) \to f(x_1,\ldots,x_i,\ldots,x_n)$$
$$f(x_1,\ldots,\mathsf{mark}(x_i),\ldots,x_n) \to f(x_1,\ldots,x_i,\ldots,x_n)$$

we obtain $\mathsf{active}(f(t_1,\ldots,t_n)) \xrightarrow{\mathsf{i}}{}^*_3 \mathsf{active}(f(s_1,\ldots,s_n))$, and hence $\mathsf{mark}(s){\downarrow_\mathcal{M}} \xrightarrow{\mathsf{i}}{}^*_3 \mathsf{active}(s)$ as desired. ☐

Now we can prove the soundness of Θ_3 for innermost termination.

Theorem 18. *Let $(\mathcal{R},\mu)$ be a CSRS. If $\mathcal{R}^3_\mu$ is innermost terminating then $(\mathcal{R},\mu)$ is innermost terminating.*

Proof. The proof is similar to the soundness proof of Θ_1 (Theorem 5), but there are also some crucial differences. Let $\mathcal{F}$ be the signature of $\mathcal{R}$. To prove the soundness of Θ_1, we showed that for all $s,t \in \mathcal{T}(\mathcal{F},\mathcal{V})$, $s \xrightarrow{\mathsf{i}}_\mu t$ implies $\mathsf{mark}(s\theta){\downarrow_\mathcal{M}} \xrightarrow{\mathsf{i}}{}^+_1 \mathsf{mark}(t\theta){\downarrow_\mathcal{M}}$. Here θ substitutes all variables by an arbitrary constant c from $\mathcal{F}$.[7] In contrast, we now show that $s \xrightarrow{\mathsf{i}}_\mu t$ implies $\mathsf{mark}(s){\downarrow_\mathcal{M}} \xrightarrow{\mathsf{i}}{}^+_3 \mathsf{mark}(t){\downarrow_\mathcal{M}}$. In general, $\mathsf{mark}(s\theta){\downarrow_\mathcal{M}} \xrightarrow{\mathsf{i}}{}^+ \mathsf{mark}(t\theta){\downarrow_\mathcal{M}}$ holds for $\mathcal{R}^1_\mu$, but not for $\mathcal{R}^3_\mu$ and $\mathsf{mark}(s){\downarrow_\mathcal{M}} \xrightarrow{\mathsf{i}}{}^+ \mathsf{mark}(t){\downarrow_\mathcal{M}}$ holds for $\mathcal{R}^3_\mu$, but not for $\mathcal{R}^1_\mu$. So the soundness proofs of the two transformations are really different.

[7] This proof relied on the fact that $\mathsf{mark}(u\theta){\downarrow_\mathcal{M}} \xrightarrow{\mathsf{i}}{}^*_1 \mathsf{active}(u\theta)$ for all $u \in \mathcal{T}(\mathcal{F},\mathcal{V})$. However, in order to reduce $\mathsf{mark}(u\theta){\downarrow_\mathcal{M}}$ to $\mathsf{active}(u\theta)$, one has to reduce subterms $\mathsf{active}(u_i\theta)$ in a term $f(\ldots,\mathsf{active}(u_i\theta),\ldots)$ to $u_i\theta$. In $\mathcal{R}^1_\mu$ this is an innermost step, but in $\mathcal{R}^3_\mu$ this is not the case if $u_i\theta$ is an $(\mathcal{R},\mu)$-redex. For that reason we now use Lemma 17 instead. Thus, in the present proof we have to transform the reduction step $s \xrightarrow{\mathsf{i}}_\mu t$ into an $\mathcal{R}^3_\mu$-reduction step where active arguments below the redex are in $(\mathcal{R},\mu)$-normal form. Consequently, we may not apply a substitution θ to s any more, since $s \xrightarrow{\mathsf{i}}_\mu t$ does not imply that the context-sensitive reduction $s\theta \to_\mu t\theta$ is innermost.

If $s \xrightarrow{\mathrm{i}}_{\mu} t$ then there is a rule $l \to r \in \mathcal{R}$, a substitution σ, and an active position π in s such that $s|_{\pi} = l\sigma$ and $t = s[r\sigma]_{\pi}$. We prove the lemma by induction on π. If $\pi = \epsilon$ then $s = l\sigma$ and $t = r\sigma$. Since the step from s to t is innermost, all active arguments of s are in μ-normal form. Hence we can apply Lemma 17 to s, which yields $\mathsf{mark}(s){\downarrow_{\mathcal{M}}} \xrightarrow{\mathrm{i}}{}^{*}_{3} \mathsf{active}(s)$. Since $\mathsf{active}(s) \to \mathsf{mark}(t)$ is an instance of a rule in $\mathcal{R}^3_{\mu}$, we have $\mathsf{active}(s) \xrightarrow{\mathrm{i}}_3 \mathsf{mark}(t)$. We also have $\mathsf{mark}(t) \xrightarrow{\mathrm{i}}{}^{*}_{3} \mathsf{mark}(t){\downarrow_{\mathcal{M}}}$. Combining all reductions yields $\mathsf{mark}(s){\downarrow_{\mathcal{M}}} \xrightarrow{\mathrm{i}}{}^{+}_{3} \mathsf{mark}(t){\downarrow_{\mathcal{M}}}$.

If $\pi = i\pi'$ then $s = f(s_1, \ldots, s_i, \ldots, s_n)$ and $t = f(s_1, \ldots, t_i, \ldots, s_n)$ with $s_i \xrightarrow{\mathrm{i}}_{\mu} t_i$. Note that we have $i \in \mu(f)$ due to the definition of context-sensitive rewriting. For $1 \leqslant j \leqslant n$ define $s'_j = \mathsf{mark}(s_j){\downarrow_{\mathcal{M}}}$ if $j \in \mu(f)$ and $s'_j = s_j$ if $j \notin \mu(f)$. The induction hypothesis yields $s'_i \xrightarrow{\mathrm{i}}{}^{+}_{3} \mathsf{mark}(t_i){\downarrow_{\mathcal{M}}}$. The result follows since $\mathsf{mark}(s){\downarrow_{\mathcal{M}}} = \mathsf{active}(f(s'_1, \ldots, s'_i, \ldots, s'_n))$ and $\mathsf{mark}(t){\downarrow_{\mathcal{M}}} = \mathsf{active}(f(s'_1, \ldots, \mathsf{mark}(t_i){\downarrow_{\mathcal{M}}}, \ldots, s'_n))$. □

The structure of the completeness proof is similar to the proof that (full, i.e. non-innermost) termination of a CSRS $(\mathcal{R}, \mu)$ implies innermost termination of $\mathcal{R}^1_{\mu}$ (Theorem 9). In Lemma 20 we first show the result for a special set of terms T, which includes all terms that are reachable from terms of the form $\mathsf{mark}(t)$ with $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ by innermost $\mathcal{R}^3_{\mu}$-rewrite steps. Afterwards we extend this result to arbitrary terms in Theorem 21.

Definition 19. *A position π in a term $t \in \mathcal{T}(\mathcal{F}_1, \mathcal{V})$ is* activated *if either* $\mathrm{root}(t) \in \{\mathsf{active}, \mathsf{mark}\}$ *or* $\mathrm{root}(t) \in \mathcal{F}$ *and there is a* mark*-symbol at a position above π or an* active*-symbol at the position directly above π. Let T be the set consisting of all terms $t \in \mathcal{T}(\mathcal{F}_1, \mathcal{V})$ that satisfy the following properties:*

(a) mark *and* active *only occur on active positions,*
(b) mark *does not occur above* active *or* mark*,*
(c) if an active position π in t is not activated then $t|_{\pi}$ is not an $\mathcal{R}$-redex,
(d) if π is an activated position in t, then all positions above π are also activated.

Here, the argument positions of active *and* mark *are also considered active.*

Lemma 20. *Let $(\mathcal{R}, \mu)$ be an innermost terminating CSRS. All terms in T are innermost $\mathcal{R}^3_{\mu}$-terminating.*

Proof. Let $\mathcal{F}$ be the signature of $\mathcal{R}$. We first show that $t \xrightarrow{\mathrm{i}}_3 u$ and $t \in T$ imply $u \in T$. For that purpose we consider the different forms of rules in $\mathcal{R}^3_{\mu}$ that can be used in the reduction step from t to u. Let π be the position of the redex contracted in $t \xrightarrow{\mathrm{i}}_3 u$. Note that to prove conditions (c) and (d) for the term u, it is sufficient only to consider positions below π. The reason is that the context surrounding $u|_{\pi}$ is unchanged in the reduction step from t to u and, due to condition (d), π and all positions above π are always activated.

1. First we regard the case where $t|_{\pi} = \mathsf{active}(l\sigma)$ and $u = t[\mathsf{mark}(r\sigma)]_{\pi}$. Since the reduction step from t to u is innermost, $l\sigma$ cannot contain any $\mathcal{R}^3_{\mu}$-redex.

As $\mathrm{root}(l) \in \mathcal{F}$, this implies that $l\sigma$ does not contain any active or mark-symbols. Hence this is also true for $r\sigma$. Consequently, u inherits properties (a) and (b) from t. Since all positions below π in u have a mark-symbol above them (at position π), u satisfies also properties (c) and (d).

2. Now let $t|_\pi = \mathsf{mark}(f(t_1,\ldots,t_n))$ and $u = t[\mathsf{active}(f([t_1]^f_1,\ldots,[t_n]^f_n))]_\pi$. Since t satisfies properties (a) and (b), u satisfies these properties, too. Since all active positions in the subterms $t_1,\ldots,t_n$ of u have a mark-symbol above them, u satisfies property (c). For property (d) we observe that in u, the positions in t_i for $i \notin \mu(f)$ are not activated (since $t_1,\ldots,t_n \in \mathcal{T}(\mathcal{F},\mathcal{V})$, as t satisfies property (b)).
3. Next we regard the case where $t|_\pi = f(t_1,\ldots,\mathsf{active}(t_i),\ldots,t_n)$ and $u = t[f(t_1,\ldots,t_i,\ldots,t_n)]_\pi$. The term u clearly satisfies properties (a) and (b). In order to conclude property (c), it suffices to show that t_i is not an $\mathcal{R}$-redex. Suppose to the contrary that t_i is an $\mathcal{R}$-redex. This implies that $\mathsf{active}(t_i) = \mathsf{active}(l\sigma) \to_3 \mathsf{mark}(r\sigma)$ for some $l \to r \in \mathcal{R}$ and substitution σ, which contradicts the assumption that the reduction step from t to u is innermost. We conclude that u satisfies property (c). The term u satisfies also property (d), because if t_i contains active or mark, then there cannot be a function symbol from $\mathcal{F}$ above it (otherwise the reduction step is not innermost).
4. Finally, we consider the case where $t|_\pi = f(t_1,\ldots,\mathsf{mark}(t_i),\ldots,t_n)$ and $u = t[f(t_1,\ldots,t_i,\ldots,t_n)]_\pi$. Since the step from t to u is innermost and t_i does not contain active or mark-symbols according to property (b), t_i must be a variable. But then it trivially follows that u inherits the four properties of t.

Let $\mathrm{erase}\colon \mathcal{T}(\mathcal{F}_1,\mathcal{V}) \to \mathcal{T}(\mathcal{F},\mathcal{V})$ remove all active and mark-symbols, i.e.,

$$\begin{aligned}
\mathrm{erase}(x) &= x && \text{for all variables } x\\
\mathrm{erase}(f(t_1,\ldots,t_n)) &= f(\mathrm{erase}(t_1),\ldots,\mathrm{erase}(t_n)) && \text{for all } f \in \mathcal{F}\\
\mathrm{erase}(\mathsf{active}(t)) &= \mathrm{erase}(\mathsf{mark}(t)) = \mathrm{erase}(t)
\end{aligned}$$

We want to transform every infinite innermost $\mathcal{R}^3_\mu$-reduction of a term $t \in T$ into an infinite innermost context-sensitive reduction of $\mathrm{erase}(t)$. Let $\mathcal{M}'$ be the subset of $\mathcal{R}^3_\mu$ consisting of $\mathcal{M}$ together with all rules of the form

$$\begin{aligned}
f(x_1,\ldots,\mathsf{active}(x_i),\ldots,x_n) &\to f(x_1,\ldots,x_i,\ldots,x_n)\\
f(x_1,\ldots,\mathsf{mark}(x_i),\ldots,x_n) &\to f(x_1,\ldots,x_i,\ldots,x_n)
\end{aligned}$$

Clearly $t \to_{\mathcal{M}'} u$ implies $\mathrm{erase}(t) = \mathrm{erase}(u)$. Since $\mathcal{M}'$ is terminating (which is shown by RPO using the precedence $\mathsf{mark} > \mathsf{active}$), every infinite $\mathcal{R}^3_\mu$-reduction contains infinitely many reduction steps with rules from $\mathcal{R}^3_\mu \setminus \mathcal{M}'$. We now show that for all $t \in T$, if $t \xrightarrow{\mathrm{i}}_3 u$ by applying a rule from $\mathcal{R}^3_\mu \setminus \mathcal{M}'$ then $\mathrm{erase}(t) \xrightarrow{\mathrm{i}}_\mu \mathrm{erase}(u)$. Thus, every infinite innermost $\mathcal{R}^3_\mu$-reduction starting from T can be transformed into an infinite reduction in $(\mathcal{R},\mu)$, which proves the lemma.

There exist a position π in t, a rewrite rule $l \to r \in \mathcal{R}$, and a substitution σ such that $t|_\pi = \mathsf{active}(l\sigma)$ and $u = t[\mathsf{mark}(r\sigma)]_\pi$. In case 1 above we already observed that $l\sigma$ and $r\sigma$ belong to $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Hence $\text{erase}(t) = \text{erase}(t)[l\sigma]_{\pi'}$ and $\text{erase}(u) = \text{erase}(u)[r\sigma]_{\pi'}$ for some position π' which is active (since π is active in t due to property (a) in the definition of T). Therefore, $\text{erase}(t) \to_\mu \text{erase}(u)$. It remains to show that this is really an innermost context-sensitive rewrite step. Suppose that $l\sigma$ contains an $\mathcal{R}$-redex on an active position $\pi'' > \epsilon$. Then this $\mathcal{R}$-redex occurs in $t = t[\mathsf{active}(l\sigma)]_\pi$ at the active position $\pi 1 \pi''$. According to property (c), this position has to be activated, which means that there is a mark-symbol above it or an active-symbol directly above it. Since $l\sigma \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ the second alternative is impossible and the first alternative would contradict property (b). Hence we indeed have $\text{erase}(t) \xrightarrow{\mathrm{i}}_\mu \text{erase}(u)$. □

Now we can show the desired completeness result.

Theorem 21. *Let $(\mathcal{R}, \mu)$ be a CSRS. If $(\mathcal{R}, \mu)$ is innermost terminating then $\mathcal{R}^3_\mu$ is innermost terminating.*

Proof. Let $\mathcal{F}$ be the signature of $\mathcal{R}$. Suppose that $\mathcal{R}^3_\mu$ is not innermost terminating. Then there exists a minimal term $s \in \mathcal{T}(\mathcal{F}_1, \mathcal{V})$ with an infinite innermost $\mathcal{R}^3_\mu$-reduction (i.e., all proper subterms of s only have finite innermost $\mathcal{R}^3_\mu$-reductions). So every infinite innermost $\mathcal{R}^3_\mu$-reduction from s contains a root reduction step. Let $t \xrightarrow{\mathrm{i}}_3 u$ be the first such root reduction step. So all proper subterms of t admit only finite innermost $\mathcal{R}^3_\mu$-reductions. Note that we cannot have $t = f(t_1, \ldots, \mathsf{active}(t_i), \ldots, t_n)$ or $t = f(t_1, \ldots, \mathsf{mark}(t_i), \ldots, t_n)$ and $u = f(t_1, \ldots, t_n)$. The reason is that then u can only have an infinite innermost reduction if one of its subterms has an infinite innermost reduction, but this would contradict the minimality of t. If $t = \mathsf{mark}(f(t_1, \ldots, t_n))$ and $u = \mathsf{active}(f([t_1]^f_1, \ldots, [t_n]^f_n))$ then $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ as the step from t to u is innermost and $f \in \mathcal{F}$. Thus, $t \in T$. But since all terms in T are innermost $\mathcal{R}^3_\mu$-terminating by Lemma 20 this is impossible. So $t = \mathsf{active}(l\sigma)$ and $u = \mathsf{mark}(r\sigma)$ for some rule $l \to r \in \mathcal{R}$ and substitution σ. We again infer that $l\sigma$ and $r\sigma$ belong to $\mathcal{T}(\mathcal{F}, \mathcal{V})$, and thus we obtain $u \in T$ which contradicts Lemma 20. Hence $\mathcal{R}^3_\mu$ is innermost terminating. □

To demonstrate the use of Θ_3, in Appendix A we show for several CSRS $(\mathcal{R}, \mu)$ including Example 1 how innermost termination of $\mathcal{R}^3_\mu$ can be proved with dependency pairs.

6 Ground Innermost Termination

Unlike for termination, to conclude innermost termination it is not sufficient to prove that all ground terms are innermost terminating.

Example 22. This is witnessed by the TRS $\{\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{f}(x)), \mathsf{f}(\mathsf{a}) \to \mathsf{a}\}$. This TRS is not innermost terminating but ground innermost terminating over the signature $\{\mathsf{f}, \mathsf{a}\}$, i.e., all ground terms only permit finite innermost reductions.

It is well known that innermost termination of a TRS $\mathcal{R}$ over a signature $\mathcal{F}$ is equivalent to ground innermost termination of $\mathcal{R}$ over the signature $\mathcal{F} \cup \{\mathsf{c}, \mathsf{h}\}$ where c is a fresh constant and h is a fresh unary function symbol. The reason is that a term t with the variables $x_1, \ldots, x_n$ starts an infinite innermost reduction iff the ground term $t\sigma$ starts an infinite innermost reduction where $\sigma(x_i) = \mathsf{h}^i(\mathsf{c})$. So the fresh symbols c and h are needed to create arbitrarily many different ground terms (in order to handle non-linear rewrite rules). A similar correspondence holds for innermost *context-sensitive* reductions with $\mu(\mathsf{h}) = \varnothing$ or $\mu(\mathsf{h}) = \{1\}$.

The following results state that Θ_1 and Θ_2 cannot distinguish ground innermost termination from innermost termination. This provides further explanation for the incompleteness of these transformation for innermost termination. Because Θ_1 and Θ_2 are sound for innermost termination, it follows that they are sound for ground innermost termination, too.

Theorem 23. *Let $(\mathcal{R}, \mu)$ be a CSRS. The TRS $\mathcal{R}^1_\mu$ is ground innermost terminating if and only if it is innermost terminating.*

Proof. The "if" direction is trivial. For the "only if" direction we reason as follows. Let $\mathcal{F}$ be the signature of $\mathcal{R}$, let c be any constant in $\mathcal{F}$, and let M be the set consisting of all terms $\mathsf{mark}^n(x)$ with $x \in \mathcal{V}$ and $n \geqslant 0$. For any term $t \in \mathcal{T}(\mathcal{F}_1, \mathcal{V})$ we let $\varphi(t)$ denote the result of replacing in t all maximal subterms belonging to M by c. Note that $\varphi(t) \in \mathcal{T}(\mathcal{F}_1)$. We show that if $s \xrightarrow{\mathsf{i}}_1 t$ with $s, t \in \mathcal{T}(\mathcal{F}_1, \mathcal{V})$ then $\varphi(s) \xrightarrow{\mathsf{i}}{}_1^+ \varphi(t)$. So any infinite innermost reduction gives rise to an infinite ground innermost reduction, which proves the theorem. We distinguish three cases.

1. First suppose that $s|_\pi = \Delta(\mathsf{active}(l)\sigma)$ and $t = s[\Delta(\mathsf{mark}(r)\sigma)]_\pi$ for some position π, substitution σ, rule $l \to r \in \mathcal{R}$, and sequence Δ of mark-symbols (where we ignore parentheses around function arguments) such that there is no mark-symbol directly above the position π in s. Let the substitution σ' be defined by $\sigma'(x) = \varphi(\sigma(x))$ for all variables x. Then we have

$$\begin{array}{rll} \varphi(s) = & \varphi(s)[\Delta(\mathsf{active}(l)\sigma')]_\pi & (l \text{ does not contain } \mathsf{mark}\text{-symbols}) \\ \xrightarrow{\mathsf{i}}_1 & \varphi(s)[\Delta(\mathsf{mark}(r)\sigma')]_\pi & (\mathsf{active}(l)\sigma' \text{ is an innermost redex}) \\ \xrightarrow{\mathsf{i}}{}_1^* & \varphi(s)[\varphi(\Delta(\mathsf{mark}(r)\sigma))]_\pi & (\text{see explanation below}) \\ = & \varphi(t) & \end{array}$$

   It remains to show that $\Delta(\mathsf{mark}(r)\sigma') \xrightarrow{\mathsf{i}}{}_1^* \varphi(\Delta(\mathsf{mark}(r)\sigma))$. We distinguish two cases. If $r\sigma \notin M$ then $\Delta(\mathsf{mark}(r)\sigma') = \varphi(\Delta(\mathsf{mark}(r)\sigma))$. If $r\sigma \in M$ then $\Delta(\mathsf{mark}(r)\sigma) \in M$ and $r \in \mathcal{V}$ and thus $r\sigma' = c = \varphi(\Delta(\mathsf{mark}(r)\sigma))$. An easy induction proof on the length of Δ reveals that $\Delta(\mathsf{mark}(c)) \xrightarrow{\mathsf{i}}{}_1^+ c$ and hence we are done.

2. Let $s|_\pi = \mathsf{mark}(f(u_1,\dots,u_n))$ and $t = s[\mathsf{active}(f([u_1]^f_1,\dots,[u_n]^f_n))]_\pi$ for some position π, n-ary function symbol $f \in \mathcal{F}$, and terms $u_1,\dots,u_n$. Then we have

$$\begin{aligned}
\varphi(s) = {} & \varphi(s)[\mathsf{mark}(f(\varphi(u_1),\dots,\varphi(u_n)))]_\pi \\
\xrightarrow{\mathsf{i}}_1 {} & \varphi(s)[\mathsf{active}(f([\varphi(u_1)]^f_1,\dots,[\varphi(u_n)]^f_n))]_\pi \\
\xrightarrow{\mathsf{i}}{}^*_1 {} & \varphi(s)[\mathsf{active}(f(\varphi([u_1]^f_1),\dots,\varphi([u_n]^f_n)))]_\pi \qquad \text{(see explanation below)} \\
= {} & \varphi(t)
\end{aligned}$$

We show that we always have $[\varphi(u_i)]^f_i \xrightarrow{\mathsf{i}}{}^*_1 \varphi([u_i]^f_i)$. For $i \notin \mu(f)$ this is clear, since $[\varphi(u_i)]^f_i = \varphi(u_i) = \varphi([u_i]^f_i)$. If $i \in \mu(f)$ then $[\varphi(u_i)]^f_i = \mathsf{mark}(\varphi(u_i))$ and $\varphi([u_i]^f_i) = \varphi(\mathsf{mark}(u_i))$. We distinguish two cases. If $u_i \notin M$ then $\mathsf{mark}(u_i) \notin M$ and thus $\mathsf{mark}(\varphi(u_i)) = \varphi(\mathsf{mark}(u_i))$. If $u_i \in M$ then $\mathsf{mark}(u_i) \in M$ and thus $\mathsf{mark}(\varphi(u_i)) = \mathsf{mark}(c)$ and $\varphi(\mathsf{mark}(u_i)) = c$. Since $\mathsf{mark}(c) \xrightarrow{\mathsf{i}}_1 \mathsf{active}(c) \xrightarrow{\mathsf{i}}_1 c$, the result follows.

3. Finally, let $s|_\pi = \Delta(\mathsf{active}(u))$ and $t = s[\Delta(u)]_\pi$ for some position π, term u, and Δ as in case 1 of this proof. Then we have

$$\begin{aligned}
\varphi(s) = {} & \varphi(s)[\Delta(\mathsf{active}(\varphi(u)))]_\pi \\
\xrightarrow{\mathsf{i}}_1 {} & \varphi(s)[\Delta(\varphi(u))]_\pi \qquad (\varphi(u) \text{ is a normal form})
\end{aligned}$$

and $\varphi(t) = \varphi(s)[\varphi(\Delta(u))]_\pi$. If $u \notin M$ then $\Delta(\varphi(u)) = \varphi(\Delta(u))$. If $u \in M$ then $\Delta(u) \in M$ and thus $\Delta(\varphi(u)) = \Delta(c)$ and $\varphi(\Delta(u)) = c$. It is easy to show by induction on the length of Δ that $\Delta(c) \xrightarrow{\mathsf{i}}{}^*_1 c$.

□

Theorem 24. *Let $(\mathcal{R},\mu)$ be a CSRS. The TRS $\mathcal{R}^2_\mu$ is ground innermost terminating if and only if it is innermost terminating.*

Proof. The "if" direction is trivial. For the "only if" direction suppose $\mathcal{R}^2_\mu$ is ground innermost terminating. From the proof of Theorem 7 it follows that $(\mathcal{R},\mu)$ is terminating. Since Θ_2 is complete for termination, $\mathcal{R}^2_\mu$ is terminating and thus also innermost terminating. □

Because Θ_3 is sound and complete for innermost termination, ground innermost termination of $\mathcal{R}^3_\mu$ does not imply innermost termination of $\mathcal{R}^3_\mu$ in general. In fact, Θ_3 is also sound and complete for ground innermost termination.

Theorem 25. *A CSRS $(\mathcal{R},\mu)$ is ground innermost terminating if and only if $\mathcal{R}^3_\mu$ is ground innermost terminating.*

Proof. The proofs of Theorems 18 and 21 can easily be adapted. It is worth remarking that the restriction to ground terms does not simplify the proofs significantly. The main difference is that one can immediately conclude that an innermost $\mathcal{R}^3_\mu$-redex has no mark strictly below the root if one is restricted to ground terms. □

	termination		ground innermost termination		innermost termination	
	sound	complete	sound	complete	sound	complete
Θ_{L}	✓	×	×	×	×	×
Θ_{Z}	✓	×	×	×	×	×
Θ_{FR}	✓	×	×	×	×	×
Θ_1	✓	×	✓	×	✓	×
Θ_2	✓	✓	✓	×	✓	×
Θ_3	✓	×	✓	✓	✓	✓

Fig. 2. Summary.

One might think that the ($\flat$)-marked rules in Definition 14 are not needed to obtain a sound and complete transformation for ground innermost termination. While soundness is easily proved, completeness does *not* hold.

Example 26. Consider the (ground) innermost terminating CSRS $(\mathcal{R}, \mu)$ from Example 16 again. Since the innermost cycle only involves ground terms, the transformed TRS without the ($\flat$)-marked rules is not ground innermost terminating.

As explained above, a transformation that is sound for ground innermost termination can also be used for innermost termination analysis by adding fresh function symbols to the signature. However, for completeness the situation is different. Here, it is desirable that the transformation is not only complete for ground, but also for full innermost termination. The reason is that while there do exist techniques to analyze ground innermost termination [11], the best-known technique for automated innermost termination analysis [1] really checks full (non-ground) innermost termination of TRSs. A complete transformation for innermost termination transforms every innermost terminating CSRS into an innermost terminating TRS and hence, innermost termination of this TRS can potentially be checked by every technique for innermost termination analysis of ordinary TRSs. But if the transformed TRS is only ground innermost terminating, (full) innermost termination analysis techniques for TRSs cannot be applied successfully.

7 Comparison

Figure 2 contains a summary of the soundness and completeness results covered in the preceding sections. The negative results for ground innermost termination for Θ_{L}, Θ_{Z}, and Θ_{FR} are shown by the same examples used to demonstrate the corresponding results for innermost termination, cf. the first paragraph of Section 3. The results on termination for Θ_3 follow from Theorem 34 below.

Moreover, in order to assess the relative power of our transformations, we illustrate in Figure 3 the relationship between the following twelve properties:

$$
\begin{tikzcd}
(2) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & (1) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & (6) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & (5) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & (9) \arrow[d, Leftrightarrow] \\
(4) & (3) \arrow[d, Leftrightarrow] & (10) & (8) & (12) \\
 & (7) \arrow[d, Leftrightarrow] & & & \\
 & (11) & & &
\end{tikzcd}
$$

Fig. 3. Comparison.

(1)	$(\mathcal{R}, \mu)$ is terminating	(5)	$(\mathcal{R}, \mu)$ is innermost terminating
(2)	$\mathcal{R}_\mu^1$ is terminating	(6)	$\mathcal{R}_\mu^1$ is innermost terminating
(3)	$\mathcal{R}_\mu^2$ is terminating	(7)	$\mathcal{R}_\mu^2$ is innermost terminating
(4)	$\mathcal{R}_\mu^3$ is terminating	(8)	$\mathcal{R}_\mu^3$ is innermost terminating

(9) $(\mathcal{R}, \mu)$ is ground innermost terminating
(10) $\mathcal{R}_\mu^1$ is ground innermost terminating
(11) $\mathcal{R}_\mu^2$ is ground innermost terminating
(12) $\mathcal{R}_\mu^3$ is ground innermost terminating

Implication (2) $\Rightarrow$ (1) is the soundness of transformation Θ_1 for termination [14], implication (1) $\Rightarrow$ (6) is Theorem 9, implication (6) $\Rightarrow$ (5) is Theorem 5, and implication (5) $\Rightarrow$ (9) is trivial.

Equivalence (1) $\Leftrightarrow$ (3) is the soundness and completeness of Θ_2 for termination [14], equivalence (3) $\Leftrightarrow$ (7) is Theorem 7, equivalence (10) $\Leftrightarrow$ (6) is Theorem 23, equivalence (11) $\Leftrightarrow$ (7) is Theorem 24, and equivalence (9) $\Leftrightarrow$ (12) is Theorem 25. The equivalence of (5) and (8) amounts to the soundness and completeness of transformation Θ_3 for innermost termination (Theorems 18 and 21). The equivalence of (2) and (4) means that Θ_1 and Θ_3 are equally powerful when it comes to proving termination. This may not come as a surprise but the proof, which is given below, is surprisingly difficult.

None of the missing implications in Figure 3 hold, except those that follow by transitivity: (1) $\not\Rightarrow$ (2) and (5) $\not\Rightarrow$ (6) are the incompleteness of Θ_1 for termination (Example 4) and innermost termination (Example 6). Moreover, (6) $\not\Rightarrow$ (1) follows by using $\mu(\mathsf{f}) = \{1, 2, 3\}$ in Example 4 and (9) $\not\Rightarrow$ (5) follows from Example 22 with $\mu(\mathsf{f}) = \{1\}$.

In the next few pages we prove that the transformations Θ_1 and Θ_3 are equivalent when it comes to termination. First we show that termination of $\mathcal{R}_\mu^1$ implies termination of $\mathcal{R}_\mu^3$. For termination it suffices to regard ground terms (as noted in Section 6 this is different from innermost termination). The problem when simulating $\mathcal{R}_\mu^3$-steps with $\mathcal{R}_\mu^1$ are the last rules of $\mathcal{R}_\mu^3$ which allow the elimination of mark-symbols below symbols from $\mathcal{F}$. However, for ground terms t without adjacent active or mark-symbols and without such symbols at the root, one can show $\mathsf{mark}(t) \to_1^+ t$. So instead of regarding arbitrary ground terms, our aim is to transform every reduction sequence into a reduction between terms t of this special form. More precisely, we show that every ground reduction step $s \to_3 t$ in $\mathcal{R}_\mu^3$ corresponds to a reduction $s{\downarrow_\mathcal{A}} \to_1^* t{\downarrow_\mathcal{A}}$ where $\mathcal{A}$ removes adjacent active and mark-symbols by replacing them by the rightmost such symbol. Moreover, if $s \to_3 t$ by applying a rule of the form $\mathsf{active}(l) \to \mathsf{mark}(r)$ then $s{\downarrow_\mathcal{A}} \to_1^+ t{\downarrow_\mathcal{A}}$. Since the remaining rules constitute a terminating subset of $\mathcal{R}_\mu^3$, any infinite $\mathcal{R}_\mu^3$-reduction would then give rise to an infinite $\mathcal{R}_\mu^1$-reduction.

Definition 27. *Let $\mathcal{A}$ be the rewrite system consisting of the following rules:*

$$\begin{aligned} \mathsf{active}(\mathsf{active}(x)) &\to \mathsf{active}(x) & \qquad \mathsf{active}(\mathsf{mark}(x)) &\to \mathsf{mark}(x) \\ \mathsf{mark}(\mathsf{active}(x)) &\to \mathsf{active}(x) & \qquad \mathsf{mark}(\mathsf{mark}(x)) &\to \mathsf{mark}(x) \end{aligned}$$

It is easy to see that $\mathcal{A}$ is terminating and confluent.

The following two preliminary results will come in handy. Lemma 28 states that if t contains no adjacent active and mark-symbols and $\mathrm{root}(t)$ is from $\mathcal{F}$, then $\mathsf{mark}(t)$ can always be reduced to $\mathsf{active}(t)$ in $\mathcal{R}_\mu^1$.

Lemma 28. *Let $(\mathcal{R}, \mu)$ be a CSRS over a signature $\mathcal{F}$ and let $t \in \mathcal{T}(\mathcal{F}_1)$ with $\mathrm{root}(t) \in \mathcal{F}$. If $t{\downarrow_\mathcal{A}} = t$ then $\mathsf{mark}(t) \to_1^+ \mathsf{active}(t)$.*

Proof. The lemma is proved by induction on the term structure of t. If t is a constant, then the rule $\mathsf{mark}(t) \to \mathsf{active}(t)$ is contained in $\mathcal{R}_\mu^1$. Otherwise, t has the form $f(t_1, \ldots, t_n)$ for some $f \in \mathcal{F}$. Define terms $s_1, \ldots, s_n$ as follows:

$$s_i = \begin{cases} u_i & \text{if } i \in \mu(f) \text{ and either } t_i = \mathsf{active}(u_i) \text{ or } t_i = \mathsf{mark}(u_i) \\ t_i & \text{otherwise} \end{cases}$$

Let $1 \leqslant i \leqslant n$. We claim that $t_i \to_1^* s_i$. If $s_i = t_i$ this is trivial. If $t_i = \mathsf{active}(u_i)$ and $s_i = u_i$ this follows by applying the rule $\mathsf{active}(x) \to x$. If $t_i = \mathsf{mark}(u_i)$ and $s_i = u_i$ then $\mathrm{root}(u_i) \in \mathcal{F}$ because t is an $\mathcal{A}$-normal form and hence we can apply the induction hypothesis. This yields $t_i \to_1^* \mathsf{active}(u_i)$ and thus $t_i \to_1^* u_i$ by an application of the rule $\mathsf{active}(x) \to x$. We obtain

$$\begin{aligned} \mathsf{mark}(t) &\to_1^* \mathsf{mark}(f(s_1, \ldots, s_n)) \\ &\to_1 \mathsf{active}(f([s_1]_1^f, \ldots, [s_n]_n^f)) \\ &\to_1^* \mathsf{active}(f(t_1, \ldots, t_n)) \qquad \text{(see explanation below)} \end{aligned}$$

We show that $[s_i]_i^f \to_3^* t_i$ for all $1 \leqslant i \leqslant n$.

If $i \in \mu(f)$ and $t_i = \mathsf{mark}(u_i)$ then $[s_i]_i^f = \mathsf{mark}(u_i) = t_i$. If $i \in \mu(f)$ and $t_i = \mathsf{active}(u_i)$ then $[s_i]_i^f = \mathsf{mark}(u_i) \to_1^* \mathsf{active}(u_i) = t_i$ by the induction hypothesis (which is applicable because $\mathrm{root}(u_i) \in \mathcal{F}$ due to the requirements on t). Otherwise we have $s_i = t_i$ and $\mathrm{root}(t_i) \in \mathcal{F}$. If $i \in \mu(f)$ then

$[s_i]_i^f = \mathsf{mark}(s_i) \to_1^* \mathsf{active}(s_i) \to_1 s_i = t_i$ by the induction hypothesis and an application of the rule $\mathsf{active}(x) \to x$. If $i \notin \mu(f)$ then $[s_i]_i^f = s_i = t_i$. ☐

The next lemma shows how to eliminate active or mark-symbols at the root of terms by $\mathcal{R}_\mu^1$-reductions. Together with Lemma 28 this implies $\mathsf{mark}(t) \to_1^+ t$ for ground terms t with $\mathrm{root}(t) \in \mathcal{F}$ and without adjacent active or mark-symbols. Hence, for such (sub)terms, the last rules of $\mathcal{R}_\mu^3$ can also be simulated in $\mathcal{R}_\mu^1$.

Lemma 29. *For every $t \in \mathcal{T}(\mathcal{F}_1)$, $\mathsf{active}(t){\downarrow_\mathcal{A}} \to_1^* t{\downarrow_\mathcal{A}}$ and $\mathsf{mark}(t){\downarrow_\mathcal{A}} \to_1^* t{\downarrow_\mathcal{A}}$.*

Proof. We distinguish three cases. If $\mathrm{root}(t) \in \mathcal{F}$ then $\mathsf{active}(t){\downarrow_\mathcal{A}} = \mathsf{active}(t{\downarrow_\mathcal{A}}) \to_1 t{\downarrow_\mathcal{A}}$ and $\mathsf{mark}(t){\downarrow_\mathcal{A}} = \mathsf{mark}(t{\downarrow_\mathcal{A}}) \to_1^+ \mathsf{active}(t{\downarrow_\mathcal{A}}) \to_1 t{\downarrow_\mathcal{A}}$ by Lemma 28. Let Δ denote an arbitrary sequence of active and mark-symbols. If $t = \Delta(\mathsf{active}(u))$ and $\mathrm{root}(u) \in \mathcal{F}$, then $\mathsf{active}(t){\downarrow_\mathcal{A}} = \mathsf{mark}(t){\downarrow_\mathcal{A}} = \mathsf{active}(u{\downarrow_\mathcal{A}}) = t{\downarrow_\mathcal{A}}$. In the remaining case we have $t = \Delta(\mathsf{mark}(u))$ with Δ and u as before, and therefore $\mathsf{active}(t){\downarrow_\mathcal{A}} = \mathsf{mark}(t){\downarrow_\mathcal{A}} = \mathsf{mark}(u{\downarrow_\mathcal{A}}) = t{\downarrow_\mathcal{A}}$. ☐

Using the two previous lemmata, we can now show that $\mathcal{R}_\mu^3$ is not more powerful than $\mathcal{R}_\mu^1$ for proving termination of CSRSs.

Theorem 30. *Let $(\mathcal{R}, \mu)$ be a CSRS. If $\mathcal{R}_\mu^1$ is terminating then $\mathcal{R}_\mu^3$ is terminating.*

Proof. Let $\mathcal{F}$ be the signature of $\mathcal{R}$. We show that if $s \to_3 t$ with $s, t \in \mathcal{T}(\mathcal{F}_1)$ then $s{\downarrow_\mathcal{A}} \to_1^* t{\downarrow_\mathcal{A}}$. Moreover, if $s \to_3 t$ by applying a rule of the form $\mathsf{active}(l) \to \mathsf{mark}(r)$ then $s{\downarrow_\mathcal{A}} \to_1^+ t{\downarrow_\mathcal{A}}$. As explained before, the remaining rules of $\mathcal{R}_\mu^3$ terminate and therefore, this proves the theorem.

1. First suppose that $s|_\pi = \Delta(\mathsf{active}(l))\sigma$ and $t = s[\Delta(\mathsf{mark}(r))\sigma]_\pi$ for some position π, substitution σ, and rule $l \to r \in \mathcal{R}$, such that there is no active or mark-symbol directly above the position π in s. Again, Δ denotes an arbitrary sequence of active and mark-symbols. Moreover, let the substitution σ' be defined by $\sigma'(x) = \sigma(x){\downarrow_\mathcal{A}}$ for all variables x. Then we have

$$
\begin{array}{rll}
s{\downarrow_\mathcal{A}} = & s{\downarrow_\mathcal{A}}[\mathsf{active}(l)\sigma']_{\pi'} & (l \text{ does not contain } \mathsf{active} \text{ or } \mathsf{mark}\text{-symbols}) \\
\to_1 & s{\downarrow_\mathcal{A}}[\mathsf{mark}(r)\sigma']_{\pi'} & \\
\to_1^* & s{\downarrow_\mathcal{A}}[\mathsf{mark}(r)\sigma{\downarrow_\mathcal{A}}]_{\pi'} & (\text{see explanation below}) \\
= & s[\mathsf{mark}(r)\sigma]_\pi{\downarrow_\mathcal{A}} & (\text{neither } \mathsf{active} \text{ nor } \mathsf{mark} \text{ directly above } \pi) \\
= & s[\Delta(\mathsf{mark}(r))\sigma]_\pi{\downarrow_\mathcal{A}} & \\
= & t{\downarrow_\mathcal{A}} &
\end{array}
$$

It remains to show that $\mathsf{mark}(r)\sigma' \to_1^* \mathsf{mark}(r)\sigma{\downarrow_\mathcal{A}}$. We distinguish three cases. If $r\sigma = \Delta'(\mathsf{active}(u))$ with $\mathrm{root}(u) \in \mathcal{F}$ then $r\sigma' = \mathsf{active}(u{\downarrow_\mathcal{A}})$ and hence

$$\begin{aligned}
\mathsf{mark}(r)\sigma' = &\ \mathsf{mark}(\mathsf{active}(u{\downarrow_{\mathcal{A}}})) \\
\to_1 &\ \mathsf{mark}(u{\downarrow_{\mathcal{A}}}) \\
\to_1^+ &\ \mathsf{active}(u{\downarrow_{\mathcal{A}}}) \qquad\qquad \text{(due to Lemma 28)} \\
= &\ \mathsf{mark}(\Delta'(\mathsf{active}(u))){\downarrow_{\mathcal{A}}} \\
= &\ \mathsf{mark}(r)\sigma{\downarrow_{\mathcal{A}}}
\end{aligned}$$

If $r\sigma = \Delta'(\mathsf{mark}(u))$ with $\mathrm{root}(u) \in \mathcal{F}$ then $r\sigma' = \mathsf{mark}(u{\downarrow_{\mathcal{A}}})$ and hence

$$\begin{aligned}
\mathsf{mark}(r)\sigma' = &\ \mathsf{mark}(\mathsf{mark}(u{\downarrow_{\mathcal{A}}})) \\
\to_1^+ &\ \mathsf{mark}(\mathsf{active}(u{\downarrow_{\mathcal{A}}})) \qquad\qquad \text{(due to Lemma 28)} \\
\to_1 &\ \mathsf{mark}(u{\downarrow_{\mathcal{A}}}) \\
= &\ \mathsf{mark}(\Delta'(\mathsf{mark}(u))){\downarrow_{\mathcal{A}}} \\
= &\ \mathsf{mark}(r)\sigma{\downarrow_{\mathcal{A}}}
\end{aligned}$$

Finally, if $\mathrm{root}(r\sigma) \in \mathcal{F}$ then we clearly have $\mathsf{mark}(r)\sigma' = \mathsf{mark}(r)\sigma{\downarrow_{\mathcal{A}}}$.

2. Let $s|_\pi = \Delta(\mathsf{mark}(f(u_1, \ldots, u_n)))$ and $t = s[\Delta(\mathsf{active}(f([u_1]_1^f, \ldots, [u_n]_n^f)))]_\pi$ for some position π, terms $u_1, \ldots, u_n$, and $f \in \mathcal{F}$, such that there is no active or mark-symbol directly above the position π in s. Then we have

$$\begin{aligned}
s{\downarrow_{\mathcal{A}}} = &\ s{\downarrow_{\mathcal{A}}}[\mathsf{mark}(f(u_1{\downarrow_{\mathcal{A}}}, \ldots, u_n{\downarrow_{\mathcal{A}}}))]_{\pi'} \\
\to_1 &\ s{\downarrow_{\mathcal{A}}}[\mathsf{active}(f([u_1{\downarrow_{\mathcal{A}}}]_1^f, \ldots, [u_n{\downarrow_{\mathcal{A}}}]_n^f))]_{\pi'} \\
\to_1^* &\ s{\downarrow_{\mathcal{A}}}[\mathsf{active}(f([u_1]_1^f{\downarrow_{\mathcal{A}}}, \ldots, [u_n]_n^f{\downarrow_{\mathcal{A}}}))]_\pi \qquad \text{(see explanation below)} \\
= &\ s[\Delta(\mathsf{active}(f([u_1]_1^f, \ldots, [u_n]_n^f)))]_\pi{\downarrow_{\mathcal{A}}} \\
= &\ t{\downarrow_{\mathcal{A}}}
\end{aligned}$$

We show that we always have $[u_i{\downarrow_{\mathcal{A}}}]_i^f \to_1^* [u_i]_i^f{\downarrow_{\mathcal{A}}}$. For $i \notin \mu(f)$ this is clear, since $[u_i{\downarrow_{\mathcal{A}}}]_i^f = u_i{\downarrow_{\mathcal{A}}} = [u_i]_i^f{\downarrow_{\mathcal{A}}}$. If $i \in \mu(f)$ then $[u_i{\downarrow_{\mathcal{A}}}]_i^f = \mathsf{mark}(u_i{\downarrow_{\mathcal{A}}})$ and $[u_i]_i^f{\downarrow_{\mathcal{A}}} = \mathsf{mark}(u_i){\downarrow_{\mathcal{A}}}$. We distinguish three cases. If $u_i = \Delta'(\mathsf{active}(u))$ with $\mathrm{root}(u) \in \mathcal{F}$ then

$$\begin{aligned}
\mathsf{mark}(u_i{\downarrow_{\mathcal{A}}}) = &\ \mathsf{mark}(\mathsf{active}(u{\downarrow_{\mathcal{A}}})) \\
\to_1 &\ \mathsf{mark}(u{\downarrow_{\mathcal{A}}}) \\
\to_1^+ &\ \mathsf{active}(u{\downarrow_{\mathcal{A}}}) \qquad\qquad \text{(by Lemma 28)} \\
= &\ \mathsf{mark}(u_i){\downarrow_{\mathcal{A}}}
\end{aligned}$$

If $u_i = \Delta'(\mathsf{mark}(u))$ with $\mathrm{root}(u) \in \mathcal{F}$ then

$$\begin{aligned}
\mathsf{mark}(u_i{\downarrow_{\mathcal{A}}}) = &\ \mathsf{mark}(\mathsf{mark}(u{\downarrow_{\mathcal{A}}})) \\
\to_1^+ &\ \mathsf{mark}(\mathsf{active}(u{\downarrow_{\mathcal{A}}})) \qquad \text{(by Lemma 28)} \\
\to_1 &\ \mathsf{mark}(u{\downarrow_{\mathcal{A}}}) \\
= &\ \mathsf{mark}(u_i){\downarrow_{\mathcal{A}}}
\end{aligned}$$

Finally, if $\mathrm{root}(u_i) \in \mathcal{F}$ then clearly $\mathsf{mark}(u_i{\downarrow_{\mathcal{A}}}) = \mathsf{mark}(u_i){\downarrow_{\mathcal{A}}}$.

3. Next let $s|_\pi = f(u_1, \ldots, \mathsf{mark}(u_i), \ldots, u_n)$ and $t = s[f(u_1, \ldots, u_n)]_\pi$ for some position π, terms $u_1, \ldots, u_n$, and $f \in \mathcal{F}$. Then we have

$$\begin{aligned} s{\downarrow_\mathcal{A}} = &\; s{\downarrow_\mathcal{A}}[f(u_1{\downarrow_\mathcal{A}}, \ldots, \mathsf{mark}(u_i){\downarrow_\mathcal{A}}, \ldots, u_n{\downarrow_\mathcal{A}})]_{\pi'} \\ \to_1^* &\; s{\downarrow_\mathcal{A}}[f(u_1{\downarrow_\mathcal{A}}, \ldots, u_i{\downarrow_\mathcal{A}}, \ldots, u_n{\downarrow_\mathcal{A}})]_{\pi'} \qquad \text{(Lemma 29)} \\ = &\; t{\downarrow_\mathcal{A}} \end{aligned}$$

4. Finally, let $s|_\pi = f(u_1, \ldots, \mathsf{active}(u_i), \ldots, u_n)$ and $t = s[f(u_1, \ldots, u_n)]_\pi$ for some position π, terms $u_1, \ldots, u_n$, and $f \in \mathcal{F}$. Then we have

$$\begin{aligned} s{\downarrow_\mathcal{A}} = &\; s{\downarrow_\mathcal{A}}[f(u_1{\downarrow_\mathcal{A}}, \ldots, \mathsf{active}(u_i){\downarrow_\mathcal{A}}, \ldots, u_n{\downarrow_\mathcal{A}})]_{\pi'} \\ \to_1^* &\; s{\downarrow_\mathcal{A}}[f(u_1{\downarrow_\mathcal{A}}, \ldots, u_i{\downarrow_\mathcal{A}}, \ldots, u_n{\downarrow_\mathcal{A}})]_{\pi'} \qquad \text{(Lemma 29)} \\ = &\; t{\downarrow_\mathcal{A}} \end{aligned}$$

□

Next we show that termination of $\mathcal{R}_\mu^3$ implies termination of $\mathcal{R}_\mu^1$. The problem when simulating $\mathcal{R}_\mu^1$-steps in $\mathcal{R}_\mu^3$ is that $\mathcal{R}_\mu^3$ does not allow the elimination of active unless there is a symbol from $\mathcal{F}$ directly above it. Thus, our aim is again to restrict ourselves to ground terms without adjacent active or mark-symbols.

We show that every ground rewrite step $s \to_1 t$ can be transformed into a reduction $\mathsf{active}(s){\downarrow_\mathcal{B}} \to_3^* \mathsf{active}(t){\downarrow_\mathcal{B}}$. Moreover, if the step $s \to_1 t$ is done by a rule of the form $\mathsf{active}(l) \to \mathsf{mark}(r)$ then $\mathsf{active}(s){\downarrow_\mathcal{B}} \to_3^+ \mathsf{active}(t){\downarrow_\mathcal{B}}$. (This is sufficient to transform infinite $\mathcal{R}_\mu^1$-reductions into infinite $\mathcal{R}_\mu^3$-reductions.) Here $\mathcal{B}$ replaces every sequence Δ of adjacent active and mark-symbols by mark, if Δ contains any mark-symbol, and by active, otherwise. Moreover, mark-symbols are propagated downwards to active positions using the rules of $\mathcal{M}$. Hence, $\mathsf{active}(s){\downarrow_\mathcal{B}}$ contains no mark-symbols and it has an active-symbol directly above every active position of s and directly above those positions which were considered active due to the active and mark-symbols in s. Thus, we use the rewrite system defined below.

Definition 31. *Let $\mathcal{B}$ be the rewrite system consisting of the rules of $\mathcal{M}$ together with the following rules:*

$$\begin{array}{rcl} \mathsf{active}(\mathsf{active}(x)) \to \mathsf{active}(x) & \qquad & \mathsf{active}(\mathsf{mark}(x)) \to \mathsf{mark}(x) \\ \mathsf{mark}(\mathsf{active}(x)) \to \mathsf{mark}(x) & \qquad & \mathsf{mark}(\mathsf{mark}(x)) \to \mathsf{mark}(x) \end{array}$$

It is easy to show that $\mathcal{B}$ is terminating and confluent.

We start with two preliminary lemmata. Lemma 32 shows that for certain terms, the result of normalizing with $\mathcal{B}$ can also be achieved with the rules of $\mathcal{R}_\mu^3$.

Lemma 32. *Let $t \in \mathcal{T}(\mathcal{F}_1)$ be in $\mathcal{B}$-normal form. If $\mathrm{root}(t) \in \mathcal{F}$ then $\mathsf{mark}(t) \to_3^* \mathsf{mark}(t){\downarrow_\mathcal{B}}$.*

Proof. The lemma is proved by induction on the term structure of t. Write $t = f(t_1,\dots,t_n)$. We define terms $s_1,u_1,\dots,s_n,u_n$ as follows:

$$s_i = \begin{cases} t_i' & \text{if } i \in \mu(f) \text{ and } t_i = \mathsf{active}(t_i') \\ t_i & \text{otherwise} \end{cases}$$

and

$$u_i = \begin{cases} \mathsf{mark}(s_i){\downarrow_{\mathcal{B}}} & \text{if } i \in \mu(f) \\ s_i & \text{otherwise} \end{cases}$$

We have

$$\begin{aligned} \mathsf{mark}(t) &\to_3^* \mathsf{mark}(f(s_1,\dots,s_n)) && \text{(see explanation below)} \\ &\to_3 \mathsf{active}(f([s_1]_1^f,\dots,[s_n]_n^f)) \\ &\to_3^* \mathsf{active}(f(u_1,\dots,u_n)) && \text{(see explanation below)} \end{aligned}$$

The initial reduction $\mathsf{mark}(t) \to_3^* \mathsf{mark}(f(s_1,\dots,s_n))$ is obtained by applications of rules of the form $f(x_1,\dots,\mathsf{active}(x_i),\dots,x_n) \to f(x_1,\dots,x_n)$. For the final part of the above reduction it is sufficient to show $[s_i]_i^f \to_3^* u_i$ for all $1 \leqslant i \leqslant n$. If $i \in \mu(f)$ and $t_i = \mathsf{active}(t_i')$ then $s_i = t_i'$ starts with a function symbol of $\mathcal{F}$ (because t_i is in $\mathcal{B}$-normal form) and thus we can apply the induction hypothesis which yields $[s_i]_i^f = \mathsf{mark}(s_i) \to_3^* \mathsf{mark}(s_i){\downarrow_{\mathcal{B}}} = u_i$. If $i \in \mu(f)$ and $\mathrm{root}(t_i) \neq \mathsf{active}$ then $s_i = t_i$ and $\mathrm{root}(t_i) \in \mathcal{F}$ (because ground $\mathcal{B}$-normal forms do not contain any mark-symbols) and thus $[s_i]_i^f = \mathsf{mark}(s_i) \to_3^* \mathsf{mark}(s_i){\downarrow_{\mathcal{B}}} = u_i$ by the induction hypothesis. If $i \notin \mu(f)$ then $[s_i]_i^f = s_i = t_i = u_i$.

Obviously, $\mathsf{active}(f(u_1,\dots,u_n))$ is in $\mathcal{B}$-normal form. In order to conclude that $\mathsf{active}(f(u_1,\dots,u_n))$ is the $\mathcal{B}$-normal form of $\mathsf{mark}(t)$, it suffices to show $\mathsf{mark}(t) \to_{\mathcal{B}}^* \mathsf{active}(f(u_1,\dots,u_n))$. We have $\mathsf{mark}(t) \to_{\mathcal{B}} \mathsf{active}(f([t_1]_1^f,\dots,[t_n]_n^f))$ and $[s_i]_i^f \to_{\mathcal{B}}^* u_i$ for all $1 \leqslant i \leqslant n$. Hence, it remains to show that $[t_i]_i^f \to_{\mathcal{B}}^* [s_i]_i^f$ for all $1 \leqslant i \leqslant n$. If $i \in \mu(f)$ and $t_i = \mathsf{active}(t_i')$ then we have $[t_i]_i^f = \mathsf{mark}(\mathsf{active}(t_i')) \to_{\mathcal{B}} \mathsf{mark}(t_i') = [s_i]_i^f$. Otherwise $t_i = s_i$ and thus $[t_i]_i^f = [s_i]_i^f$. $\square$

Lemma 33 proves that the $\mathcal{R}_\mu^3$-reduction sketched in Lemma 32 can be extended to obtain the root symbol active.

Lemma 33. *Let* $t \in \mathcal{T}(\mathcal{F}_1)$ *with* $\mathrm{root}(t) \in \mathcal{F}$. *If* $t{\downarrow_{\mathcal{B}}} = t$ *then* $\mathsf{mark}(t){\downarrow_{\mathcal{B}}} \to_3^* \mathsf{active}(t)$.

Proof. We again use induction on the term structure of t. Write $t = f(t_1,\dots,t_n)$. We define terms $u_1,v_1,\dots,u_n,v_n$ as follows:

$$u_i = \begin{cases} \mathsf{mark}(t_i'){\downarrow_{\mathcal{B}}} & \text{if } i \in \mu(f) \text{ and } t_i = \mathsf{active}(t_i') \\ \mathsf{mark}(t_i){\downarrow_{\mathcal{B}}} & \text{if } i \in \mu(f) \text{ and } \mathrm{root}(t_i) \in \mathcal{F} \\ t_i & \text{otherwise} \end{cases}$$

and

$$v_i = \begin{cases} \mathsf{active}(t_i') & \text{if } i \in \mu(f) \text{ and } t_i = \mathsf{active}(t_i') \\ \mathsf{active}(t_i) & \text{if } i \in \mu(f) \text{ and } \mathrm{root}(t_i) \in \mathcal{F} \\ t_i & \text{otherwise} \end{cases}$$

Let $1 \leqslant i \leqslant n$. We claim that $u_i \to_3^* v_i$. For $i \notin \mu(f)$ this is obvious. If $i \in \mu(f)$ and $\mathrm{root}(t_i) \in \mathcal{F}$ then $u_i = \mathsf{mark}(t_i){\downarrow_\mathcal{B}} \to_3^* \mathsf{active}(t_i) = v_i$ by the induction hypothesis. If $i \in \mu(f)$ and $t_i = \mathsf{active}(t_i')$ then $\mathrm{root}(t_i') \in \mathcal{F}$ because t is in $\mathcal{B}$-normal form. Hence we obtain $u_i \to_3^* v_i$ as in the previous case. Using this observation, now we can prove the lemma. We have

$$\begin{aligned} \mathsf{mark}(t){\downarrow_\mathcal{B}} &= \mathsf{active}(f(u_1,\dots,u_n)) && \text{(as in the proof of Lemma 32)} \\ &\to_3^* \mathsf{active}(f(v_1,\dots,v_n)) \\ &\to_3^* \mathsf{active}(f(t_1,\dots,t_n)) \\ &= \mathsf{active}(t) \end{aligned}$$

The final part of the above reduction follows by suitable applications of rules of the form $f(x_1,\dots,\mathsf{active}(x_i),\dots,x_n) \to f(x_1,\dots,x_n)$. □

With the two previous lemmata we can now prove the desired theorem.

Theorem 34. *Let $(\mathcal{R},\mu)$ be a CSRS. If $\mathcal{R}_\mu^3$ is terminating then $\mathcal{R}_\mu^1$ is terminating.*

Proof. Let $\mathcal{F}$ be the signature of $\mathcal{R}$. We claim that for terms $s,t \in \mathcal{T}(\mathcal{F}_1)$, if $s \to_1 t$ then $\mathsf{active}(s){\downarrow_\mathcal{B}} \to_3^* \mathsf{active}(t){\downarrow_\mathcal{B}}$. Moreover, if a rule of the form $\mathsf{active}(l) \to \mathsf{mark}(r)$ is used then $\mathsf{active}(s){\downarrow_\mathcal{B}} \to_3^+ \mathsf{active}(t){\downarrow_\mathcal{B}}$. Since $\mathcal{M} \cup \{\mathsf{active}(x) \to x\}$ is terminating (which can be shown by RPO using the precedence $\mathsf{mark} > \mathsf{active}$), every infinite $\mathcal{R}_\mu^1$-reduction is transformed into an infinite $\mathcal{R}_\mu^3$-reduction, which proves the theorem.

To prove the claim, we distinguish three cases depending on the form of the rewrite rule applied in $s \to_1 t$.

1. Suppose that $s|_\pi = \Delta(\mathsf{active}(l\sigma))$ and $t = s[\Delta(\mathsf{mark}(r\sigma))]_\pi$ for some position π, substitution σ, and rule $l \to r \in \mathcal{R}$, such that there is no active or mark-symbol directly above the position π in s. As usual, Δ denotes an arbitrary sequence of active and mark-symbols. Moreover, let the substitution σ' be defined by $\sigma'(x) = \sigma(x){\downarrow_\mathcal{B}}$ for all variables x. First we show that $\mathsf{active}(s){\downarrow_\mathcal{B}} \to_3^* \mathsf{active}(s){\downarrow_\mathcal{B}}[\mathsf{active}(l\sigma')]_{\underline{\pi}}$. We distinguish two cases.
   (a) Suppose $\pi = \pi'\pi''$ such that $\mathrm{root}(s|_{\pi'}) = \mathsf{mark}$ and π'' is an active position in $s|_{\pi'}$. (As usual, the argument positions of active and mark are also considered active.) In this case, when $\mathcal{B}$-normalizing $\mathsf{active}(s)$, the mark-symbol at position π' is propagated to the root of $s|_\pi$ and subsequently

consumes all active and mark-symbols in front of $l\sigma$. Hence

$$\begin{aligned}
\mathsf{active}(s){\downarrow_\mathcal{B}} = {} & \mathsf{active}(s){\downarrow_\mathcal{B}}[\mathsf{mark}(l\sigma){\downarrow_\mathcal{B}}]_{\underline{\pi}} & \\
= {} & \mathsf{active}(s){\downarrow_\mathcal{B}}[\mathsf{mark}(l\sigma'){\downarrow_\mathcal{B}}]_{\underline{\pi}} & (l\sigma \to_\mathcal{B}^* l\sigma') \\
\to_3^* {} & \mathsf{active}(s){\downarrow_\mathcal{B}}[\mathsf{active}(l\sigma')]_{\underline{\pi}} & (\text{Lemma 33})
\end{aligned}$$

Note that Lemma 33 is applicable because $l\sigma'{\downarrow_\mathcal{B}} = l\sigma'$ and $\mathrm{root}(l\sigma') \in \mathcal{F}$.

(b) If there is no mark-symbol above position π such that π is in its "active range" then we clearly have

$$\begin{aligned}
\mathsf{active}(s){\downarrow_\mathcal{B}} &= \mathsf{active}(s){\downarrow_\mathcal{B}}[\mathsf{active}(l\sigma){\downarrow_\mathcal{B}}]_{\underline{\pi}} \\
&= \mathsf{active}(s){\downarrow_\mathcal{B}}[\mathsf{active}(l\sigma')]_{\underline{\pi}}
\end{aligned}$$

It remains to prove that $\mathsf{active}(s){\downarrow_\mathcal{B}}[\mathsf{active}(l\sigma')]_{\underline{\pi}} \to_3^+ \mathsf{active}(t){\downarrow_\mathcal{B}}$. We again distinguish two cases.

(a) If $\mathrm{root}(r\sigma') \in \mathcal{F}$ then

$$\begin{aligned}
\mathsf{active}(s){\downarrow_\mathcal{B}}[\mathsf{active}(l\sigma')]_{\underline{\pi}} \to_3 {} & \mathsf{active}(s){\downarrow_\mathcal{B}}[\mathsf{mark}(r\sigma')]_{\underline{\pi}} & \\
\to_3^* {} & \mathsf{active}(s){\downarrow_\mathcal{B}}[\mathsf{mark}(r\sigma'){\downarrow_\mathcal{B}}]_{\underline{\pi}} & (\text{Lemma 32}) \\
= {} & \mathsf{active}(s){\downarrow_\mathcal{B}}[\mathsf{mark}(r\sigma){\downarrow_\mathcal{B}}]_{\underline{\pi}} & (r\sigma \to_\mathcal{B}^* r\sigma') \\
= {} & \mathsf{active}(t){\downarrow_\mathcal{B}} &
\end{aligned}$$

(b) The case where $\mathrm{root}(r\sigma') \notin \mathcal{F}$ requires some more effort. We must have $r \in \mathcal{V}$. Because $r\sigma'$ is in $\mathcal{B}$-normal form, $r\sigma' = \mathsf{active}(u)$ with $\mathrm{root}(u) \in \mathcal{F}$. We define the substitution τ as follows:

$$\tau(x) = \begin{cases} u & \text{if } x = r \\ \sigma'(x) & \text{otherwise} \end{cases}$$

By suitable applications of rules of the form $f(x_1, \ldots, \mathsf{active}(x_i), \ldots, x_n) \to f(x_1, \ldots, x_n)$ we obtain $l\sigma' \to_3^* l\tau$. We have

$$\mathsf{mark}(r\sigma') = \mathsf{mark}(\mathsf{active}(u)) \to_\mathcal{B} \mathsf{mark}(u) = \mathsf{mark}(r\tau)$$

and thus $\mathsf{mark}(r\sigma'){\downarrow_\mathcal{B}} = \mathsf{mark}(r\tau){\downarrow_\mathcal{B}}$. Therefore

$$\begin{aligned}
\mathsf{active}(s){\downarrow_\mathcal{B}}[\mathsf{active}(l\sigma')]_{\underline{\pi}} \to_3^* {} & \mathsf{active}(s){\downarrow_\mathcal{B}}[\mathsf{active}(l\tau)]_{\underline{\pi}} & \\
\to_3 {} & \mathsf{active}(s){\downarrow_\mathcal{B}}[\mathsf{mark}(r\tau)]_{\underline{\pi}} & \\
\to_3^* {} & \mathsf{active}(s){\downarrow_\mathcal{B}}[\mathsf{mark}(r\tau){\downarrow_\mathcal{B}}]_{\underline{\pi}} & (\text{Lemma 32}) \\
= {} & \mathsf{active}(s){\downarrow_\mathcal{B}}[\mathsf{mark}(r\sigma'){\downarrow_\mathcal{B}}]_{\underline{\pi}} & \\
= {} & \mathsf{active}(s){\downarrow_\mathcal{B}}[\mathsf{mark}(r\sigma){\downarrow_\mathcal{B}}]_{\underline{\pi}} & (r\sigma \to_\mathcal{B}^* r\sigma') \\
= {} & \mathsf{active}(t){\downarrow_\mathcal{B}} &
\end{aligned}$$

2. Next let $s|_\pi = \mathsf{mark}(f(u_1,\dots,u_n)))$ and $t = s[\mathsf{active}(f([u_1]^f_1,\dots,[u_n]^f_n))]_\pi$ for some position π, terms $u_1,\dots,u_n$, and $f \in \mathcal{F}$. In this case we have $s \to_{\mathcal{B}} t$ and thus trivially $\mathsf{active}(s){\downarrow_{\mathcal{B}}} = \mathsf{active}(t){\downarrow_{\mathcal{B}}}$.
3. Finally, let $s|_\pi = \Delta(\mathsf{active}(u))$ and $t = s[\Delta(u)]_\pi$ for some position π and term u, such that there is no active or mark-symbol directly above the position π in s. If Δ is not empty then $\Delta(\mathsf{active}(u)){\downarrow_{\mathcal{B}}} = \Delta(u){\downarrow_{\mathcal{B}}}$ and hence also $\mathsf{active}(s){\downarrow_{\mathcal{B}}} = \mathsf{active}(t){\downarrow_{\mathcal{B}}}$. So suppose that Δ is empty. We distinguish two further cases.
   (a) Suppose $\pi = \pi'\pi''$ such that $\mathrm{root}(s|_{\pi'}) = \mathsf{mark}$ and π'' is an active position in $s|_{\pi'}$. In this case, when $\mathcal{B}$-normalizing $\mathsf{active}(s)$, the mark-symbol at position π' is propagated to the root of $s|_\pi$ and the active-symbol at position π is subsequently consumed by an application of the rule $\mathsf{mark}(\mathsf{active}(x)) \to \mathsf{mark}(x)$ of $\mathcal{B}$. It follows that $\mathsf{active}(s){\downarrow_{\mathcal{B}}} = \mathsf{active}(s){\downarrow_{\mathcal{B}}}[\mathsf{mark}(u){\downarrow_{\mathcal{B}}}]_{\underline{\pi}} = \mathsf{active}(t){\downarrow_{\mathcal{B}}}$.
   (b) In the remaining case there is no mark-symbol above position π such that π is in its "active range". If $\pi = \epsilon$ then $\mathsf{active}(s){\downarrow_{\mathcal{B}}} = \mathsf{active}(\mathsf{active}(u)){\downarrow_{\mathcal{B}}} = \mathsf{active}(u){\downarrow_{\mathcal{B}}} = \mathsf{active}(t){\downarrow_{\mathcal{B}}}$. If $\pi > \epsilon$ then we must have $\pi = \pi' j$ with $s|_{\pi'} = f(s_1,\dots,s_n)$ and $s_j = \mathsf{active}(u)$. Hence

$$\mathsf{active}(s){\downarrow_{\mathcal{B}}} = \mathsf{active}(s){\downarrow_{\mathcal{B}}}[f(s_1{\downarrow_{\mathcal{B}}},\dots,\mathsf{active}(u){\downarrow_{\mathcal{B}}},\dots,s_n{\downarrow_{\mathcal{B}}})]_{\underline{\pi}}$$

   and

$$\mathsf{active}(t){\downarrow_{\mathcal{B}}} = \mathsf{active}(s){\downarrow_{\mathcal{B}}}[f(s_1{\downarrow_{\mathcal{B}}},\dots,u{\downarrow_{\mathcal{B}}},\dots,s_n{\downarrow_{\mathcal{B}}})]_{\underline{\pi}}$$

   If $\mathrm{root}(u) \in \{\mathsf{active},\mathsf{mark}\}$ then $\mathsf{active}(u){\downarrow_{\mathcal{B}}} = u{\downarrow_{\mathcal{B}}}$ and thus $\mathsf{active}(s){\downarrow_{\mathcal{B}}} = \mathsf{active}(t){\downarrow_{\mathcal{B}}}$. Otherwise, $\mathrm{root}(u) \in \mathcal{F}$ and thus $\mathsf{active}(u){\downarrow_{\mathcal{B}}} = \mathsf{active}(u{\downarrow_{\mathcal{B}}})$. In this latter case we apply the rewrite rule $f(x_1,\dots,\mathsf{active}(x_j),\dots,x_n) \to f(x_1,\dots,x_n)$ to conclude $\mathsf{active}(s){\downarrow_{\mathcal{B}}} \to_3 \mathsf{active}(t){\downarrow_{\mathcal{B}}}$.

$\square$

8 Conclusion

We investigated five existing transformations from context-sensitive to ordinary rewrite systems. Of these five transformations, only the transformations Θ_1 and Θ_2 from [14] are sound for proving innermost termination of CSRSs. We showed that Θ_2 is not very useful when it comes to innermost termination, but that termination of a CSRS $(\mathcal{R},\mu)$ already implies innermost termination of $\Theta_1(\mathcal{R},\mu)$. So for classes of CSRSs where termination and innermost termination are equivalent, Θ_1 is sound and complete for innermost termination. While in general Θ_1 is still incomplete, we developed a new transformation Θ_3 which is sound and complete for innermost termination. As far as (non-innermost) termination is concerned, Θ_3 and Θ_1 are equally powerful.

So with our new transformation, innermost termination of context-sensitive rewriting can be fully reduced to innermost termination of ordinary rewriting. Moreover, for orthogonal CSRSs innermost termination already suffices for termination. So for such systems, innermost termination of the transformed TRS even implies termination of the CSRS. The existing methods for *innermost* termination analysis of TRSs are much more powerful than the ones for termination. Hence, our result now enables the use of these methods for (innermost) termination of context-sensitive rewriting, cf. Appendix A, where we use our transformation in combination with the dependency pair technique for TRSs in order to verify (innermost) termination of CSRSs.

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A Examples

In this section, we demonstrate how our transformation Θ_3 can be used in combination with *dependency pairs* in order to prove innermost termination of context-sensitive rewrite systems. For an introduction to dependency pairs we refer to [1].

The TRSs $\mathcal{R}_\mu^3$ resulting from our transformation have a special form and hence, to ease their innermost termination proof, the following refinements can be used when applying dependency pairs. (Refinement (E) can even be used for arbitrary TRSs, but the other refinements are due to the special form of $\mathcal{R}^3_\mu$.)

(A) If $s \to t$ is a dependency pair with $\mathrm{root}(s) \in \{\mathsf{ACTIVE}, \mathsf{MARK}\}$, then no narrowing is needed which would instantiate variables with terms containing active or mark.

(B) If $s \to t$ is a dependency pair with $\mathrm{root}(s) \in \{\mathsf{ACTIVE}, \mathsf{MARK}\}$, then $s \to t$ can be replaced by all pairs of the form $s\mu \to w\mu$ for all dependency pairs $v \to w$ where μ is the most general unifier of $\textsc{cap}'(t)$ and v. Here, $\textsc{cap}'$ replaces all subterms built with mark or active by pairwise different fresh variables. (In other words, one can *combine* $s \to t$ with all pairs $v \to w$ which possibly follow this pair in an innermost chain.)

(C) In any dependency pair of the form

$$\mathsf{ACTIVE}(C[x]) \to C'[f(\ldots, \mathsf{mark}(x), \ldots)]$$

where x is on an active position of $C[x]$, the subterm $\mathsf{mark}(x)$ can be replaced by x, i.e., one can replace the dependency pair by

$$\mathsf{ACTIVE}(C[x]) \to C'[f(\ldots, x, \ldots)]$$

(D) If constructors only have active argument positions and $(\mathcal{R}, \mu)$ is an orthogonal constructor system[8] such that in right-hand sides of dependency pairs of $\mathcal{R}^3_\mu$ defined symbols of $\mathcal{R}$ occur only at position 1 and where dependency pairs do not contain the symbol active, then the active-rules are not "usable" [1].

(E) Rewriting dependency pairs [12] can be extended to overlapping systems as follows: if $s \to t$ is a dependency pair and $t|_\pi$ is a reducible ground term then $s \to t$ can be replaced by the pairs $s \to t[u_1]_\pi$, …, $s \to t[u_n]_\pi$, where $u_1, \ldots, u_n$ are the terms reachable from $t|_\pi$ in one innermost rewrite step.

All these refinements can also be used for modular innermost termination proofs [13] where one regards subsets of dependency pairs separately for every cycle of the innermost dependency graph.

Note that refinements (B), (C), (E), as well as the refinements of narrowing, rewriting, and instantiating dependency pairs in [12] modify the original dependency pairs to new pairs of terms. When formulating the refinements above, we also refer to these new pairs as "dependency pairs". In other words, the refinements may be applied repeatedly after each other and finally, the resulting set of pairs is taken as "the" set of dependency pairs. So for example, refinement (D) can also be applied if the set of pairs resulting from modifying the dependency pairs has the required form.

[8] A *constructor system* has the property that no defined symbol occurs below the root position in some left-hand side.

The above refinements are generally applicable when proving innermost termination of systems resulting from transforming CSRSs. The conditions for their application can be checked automatically.

We demonstrate the usefulness of our transformation with two examples. In Section A.1 we handle a variant of Example 6, i.e., a CSRS that is innermost terminating but not terminating. Example 1 (Section A.2) is a natural CSRS that is terminating but where innermost termination is significantly easier to prove than termination and where innermost termination is already sufficient for termination. A thorough justification of the refinements (A)–(E) can be found in Section A.3.

A.1 Variant of Example 6

We regard the following CSRS $(\mathcal{R}, \mu)$ with $\mathcal{R}$ consisting of the three rules

$$\mathsf{f}(\mathsf{g}(\mathsf{b})) \to \mathsf{f}(\mathsf{g}(\mathsf{a})) \qquad \mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{a}) \qquad \mathsf{a} \to \mathsf{b}$$

and $\mu(\mathsf{f}) = \{1\}$ and $\mu(\mathsf{g}) = \varnothing$. The TRS $\mathcal{R}$ is not innermost terminating. The CSRS $(\mathcal{R}, \mu)$ is innermost terminating but not terminating. (This CSRS corresponds to Example 6 extended by the additional rule $\mathsf{f}(\mathsf{g}(\mathsf{b})) \to \mathsf{f}(\mathsf{g}(\mathsf{a}))$. This rule is added to demonstrate that our method is also successful for systems which are innermost terminating as a CSRS but not as a TRS. The innermost termination proof of Example 6 proceeds in the same way.) Our transformation produces the following TRS $\mathcal{R}^3_\mu$:

$$\begin{array}{rclrcl}
\mathsf{active}(\mathsf{f}(\mathsf{g}(\mathsf{b}))) & \to & \mathsf{mark}(\mathsf{f}(\mathsf{g}(\mathsf{a}))) & \mathsf{mark}(\mathsf{f}(x)) & \to & \mathsf{active}(\mathsf{f}(\mathsf{mark}(x))) \\
\mathsf{active}(\mathsf{f}(\mathsf{a})) & \to & \mathsf{mark}(\mathsf{f}(\mathsf{a})) & \mathsf{mark}(\mathsf{g}(x)) & \to & \mathsf{active}(\mathsf{g}(x)) \\
\mathsf{active}(\mathsf{a}) & \to & \mathsf{mark}(\mathsf{b}) & \mathsf{mark}(\mathsf{a}) & \to & \mathsf{active}(\mathsf{a}) \\
 & & & \mathsf{mark}(\mathsf{b}) & \to & \mathsf{active}(\mathsf{b}) \\
\mathsf{f}(\mathsf{active}(x)) & \to & \mathsf{f}(x) & \mathsf{f}(\mathsf{mark}(x)) & \to & \mathsf{f}(x) \\
\mathsf{g}(\mathsf{active}(x)) & \to & \mathsf{g}(x) & \mathsf{g}(\mathsf{mark}(x)) & \to & \mathsf{g}(x)
\end{array}$$

We show that innermost termination of this TRS can be proved easily with dependency pairs. We omit pairs of the form $\mathsf{MARK}(\cdot) \to \mathsf{F}(\cdot)$ and $\mathsf{MARK}(\cdot) \to \mathsf{G}(\cdot)$ as well as $\mathsf{ACTIVE}(\cdot) \to \mathsf{F}(\cdot)$ and $\mathsf{ACTIVE}(\cdot) \to \mathsf{G}(\cdot)$ since these pairs are obviously not on cycles of the (estimated) innermost dependency graph. In the sequel we abbreviate MARK to M and ACTIVE to active.

$$\begin{array}{rclrrclr}
\mathsf{A}(\mathsf{f}(\mathsf{g}(\mathsf{b}))) & \to & \mathsf{M}(\mathsf{f}(\mathsf{g}(\mathsf{a}))) & (1) & \mathsf{M}(\mathsf{f}(x)) & \to & \mathsf{A}(\mathsf{f}(\mathsf{mark}(x))) & (5) \\
\mathsf{A}(\mathsf{f}(\mathsf{a})) & \to & \mathsf{M}(\mathsf{f}(\mathsf{a})) & (2) & \mathsf{M}(\mathsf{g}(x)) & \to & \mathsf{A}(\mathsf{g}(x)) & (6) \\
\mathsf{A}(\mathsf{a}) & \to & \mathsf{M}(\mathsf{b}) & (3) & \mathsf{M}(\mathsf{a}) & \to & \mathsf{A}(\mathsf{a}) & (7) \\
\mathsf{M}(\mathsf{f}(x)) & \to & \mathsf{M}(x) & (4) & \mathsf{M}(\mathsf{b}) & \to & \mathsf{A}(\mathsf{b}) & (8) \\
\mathsf{F}(\mathsf{active}(x)) & \to & \mathsf{F}(x) & (9) & \mathsf{F}(\mathsf{mark}(x)) & \to & \mathsf{F}(x) & (11) \\
\mathsf{G}(\mathsf{active}(x)) & \to & \mathsf{G}(x) & (10) & \mathsf{G}(\mathsf{mark}(x)) & \to & \mathsf{G}(x) & (12)
\end{array}$$

Dependency pairs (3), (6), (7), and (8) are not on cycles of the innermost dependency graph (this can easily be detected using refinement (B)). According to refinement (B), both (1) and (2) can be combined with dependency pairs (4) and (5) and hence are replaced by

$$\mathsf{A}(\mathsf{f}(\mathsf{g}(\mathsf{b}))) \to \mathsf{M}(\mathsf{g}(\mathsf{a})) \quad (13) \qquad \mathsf{A}(\mathsf{f}(\mathsf{a})) \to \mathsf{M}(\mathsf{a}) \quad (15)$$

$$\mathsf{A}(\mathsf{f}(\mathsf{g}(\mathsf{b}))) \to \mathsf{A}(\mathsf{f}(\mathsf{mark}(\mathsf{g}(\mathsf{a})))) \quad (14) \qquad \mathsf{A}(\mathsf{f}(\mathsf{a})) \to \mathsf{A}(\mathsf{f}(\mathsf{mark}(\mathsf{a}))) \quad (16)$$

Pairs (13) and (15) are not on cycles. Since the right-hand sides of (14) and (16) are ground, one can innermost rewrite them according to refinement (E). This yields

$$\mathsf{A}(\mathsf{f}(\mathsf{g}(\mathsf{b}))) \to \mathsf{A}(\mathsf{f}(\mathsf{g}(\mathsf{a}))) \qquad \mathsf{A}(\mathsf{f}(\mathsf{a})) \to \mathsf{A}(\mathsf{f}(\mathsf{b}))$$

With refinement (B) we immediately detect that these pairs are not on cycles and hence, they can be deleted. But then (5) is not on a cycle either, because there is no longer any dependency pair whose left-hand side has the root A. So the only dependency pairs on cycles are (4) and (9)–(12). Since these pairs have no usable rules, the resulting constraints are already satisfied by the embedding order. Hence, $\mathcal{R}^3_\mu$ is innermost terminating (and using our refinements, this innermost termination proof can easily be performed automatically).

A.2 Example 1

We regard the CSRS $(\mathcal{R}, \mu)$ with $\mathcal{R}$ consisting of the rules

$$\begin{aligned} 0 \leqslant y &\to \mathsf{true} & \mathsf{p}(0) &\to 0 \\ \mathsf{s}(x) \leqslant 0 &\to \mathsf{false} & \mathsf{p}(\mathsf{s}(x)) &\to x \\ \mathsf{s}(x) \leqslant \mathsf{s}(y) &\to x \leqslant y & \mathsf{if}(\mathsf{true}, x, y) &\to x \\ x - y &\to \mathsf{if}(x \leqslant y, 0, \mathsf{s}(\mathsf{p}(x) - y)) & \mathsf{if}(\mathsf{false}, x, y) &\to y \end{aligned}$$

with $\mu(\mathsf{if}) = \{1\}$ and $\mu(f) = \{1, \ldots, \mathrm{arity}(f)\}$ for all other function symbols f.

This system is a natural formulation of the subtraction algorithm using a conditional if. In functional languages like LISP which have no pattern matching, p and if would be built-in and one would have to formulate algorithms using if and selectors like p. A corresponding algorithm was already treated in [1, Example 41], but there the if-symbol had to be encoded in a counterintuitive way to prevent the evaluation of the third argument of if. In contrast, the formulation above is natural, but it is only possible in context-sensitive rewriting. Our transformation produces the following TRS $\mathcal{R}^3_\mu$:

$$\begin{aligned} \mathsf{active}(\mathsf{p}(0)) &\to \mathsf{mark}(0) & \mathsf{mark}(0) &\to \mathsf{active}(0) \\ \mathsf{active}(\mathsf{p}(\mathsf{s}(x))) &\to \mathsf{mark}(x) & \mathsf{mark}(\mathsf{true}) &\to \mathsf{active}(\mathsf{true}) \\ \mathsf{active}(0 \leqslant y) &\to \mathsf{mark}(\mathsf{true}) & \mathsf{mark}(\mathsf{false}) &\to \mathsf{active}(\mathsf{false}) \end{aligned}$$

$$\begin{aligned} \mathsf{active}(\mathsf{s}(x) \leqslant \mathsf{0}) &\to \mathsf{mark}(\mathsf{false}) & \mathsf{mark}(\mathsf{s}(x)) &\to \mathsf{active}(\mathsf{s}(\mathsf{mark}(x))) \\ \mathsf{active}(\mathsf{s}(x) \leqslant \mathsf{s}(y)) &\to \mathsf{mark}(x \leqslant y) & \mathsf{mark}(\mathsf{p}(x)) &\to \mathsf{active}(\mathsf{p}(\mathsf{mark}(x))) \\ \mathsf{active}(\mathsf{if}(\mathsf{true}, x, y)) &\to \mathsf{mark}(x) & \mathsf{mark}(x \leqslant y) &\to \mathsf{active}(\mathsf{mark}(x) \leqslant \mathsf{mark}(y)) \\ \mathsf{active}(\mathsf{if}(\mathsf{false}, x, y)) &\to \mathsf{mark}(y) & \mathsf{mark}(x - y) &\to \mathsf{active}(\mathsf{mark}(x) - \mathsf{mark}(y)) \end{aligned}$$

$$\begin{aligned} \mathsf{active}(x - y) &\to \mathsf{mark}(\mathsf{if}(x \leqslant y, \mathsf{0}, \mathsf{s}(\mathsf{p}(x) - y))) \\ \mathsf{mark}(\mathsf{if}(x, y, z)) &\to \mathsf{active}(\mathsf{if}(\mathsf{mark}(x), y, z)) \end{aligned}$$

$$\begin{aligned} \mathsf{s}(f(x)) &\to \mathsf{s}(x) & x \leqslant f(y) &\to x \leqslant y & \mathsf{if}(f(x), y, z) &\to \mathsf{if}(x, y, z) \\ \mathsf{p}(f(x)) &\to \mathsf{p}(x) & f(x) - y &\to x - y & \mathsf{if}(x, f(y), z) &\to \mathsf{if}(x, y, z) \\ f(x) \leqslant y &\to x \leqslant y & x - f(y) &\to x - y & \mathsf{if}(x, y, f(z)) &\to \mathsf{if}(x, y, z) \end{aligned}$$

for $f \in \{\mathsf{mark}, \mathsf{active}\}$. We show how innermost termination of this TRS is proved with dependency pairs. Since $\mathcal{R}_\mu^3$ is a non-overlapping TRS, innermost termination of this TRS coincides with its termination. Nevertheless, proving innermost termination is considerably easier than proving termination directly. We again omit dependency pairs of the form $\mathsf{M}(\cdot) \to F(\cdot)$ and $\mathsf{A}(\cdot) \to F(\cdot)$ where $f \in \mathcal{F}$ since these pairs are obviously not on cycles of the (estimated) innermost dependency graph.

$$\begin{aligned} \mathsf{A}(\mathsf{p}(\mathsf{0})) &\to \mathsf{M}(\mathsf{0}) \quad (17) & \mathsf{M}(\mathsf{0}) &\to \mathsf{A}(\mathsf{0}) \quad (24) \\ \mathsf{A}(\mathsf{p}(\mathsf{s}(x))) &\to \mathsf{M}(x) \quad (18) & \mathsf{M}(\mathsf{true}) &\to \mathsf{A}(\mathsf{true}) \quad (25) \\ \mathsf{A}(\mathsf{0} \leqslant y) &\to \mathsf{M}(\mathsf{true}) \quad (19) & \mathsf{M}(\mathsf{false}) &\to \mathsf{A}(\mathsf{false}) \quad (26) \\ \mathsf{A}(\mathsf{s}(x) \leqslant \mathsf{0}) &\to \mathsf{M}(\mathsf{false}) \quad (20) & \mathsf{M}(\mathsf{s}(x)) &\to \mathsf{A}(\mathsf{s}(\mathsf{mark}(x))) \quad (27) \\ \mathsf{A}(\mathsf{s}(x) \leqslant \mathsf{s}(y)) &\to \mathsf{M}(x \leqslant y) \quad (21) & \mathsf{M}(\mathsf{p}(x)) &\to \mathsf{A}(\mathsf{p}(\mathsf{mark}(x))) \quad (28) \\ \mathsf{A}(\mathsf{if}(\mathsf{true}, x, y)) &\to \mathsf{M}(x) \quad (22) & \mathsf{M}(x \leqslant y) &\to \mathsf{A}(\mathsf{mark}(x) \leqslant \mathsf{mark}(y)) \quad (29) \\ \mathsf{A}(\mathsf{if}(\mathsf{false}, x, y)) &\to \mathsf{M}(y) \quad (23) & \mathsf{M}(x - y) &\to \mathsf{A}(\mathsf{mark}(x) - \mathsf{mark}(y)) \quad (30) \end{aligned}$$

$$\mathsf{A}(x - y) \to \mathsf{M}(\mathsf{if}(x \leqslant y, \mathsf{0}, \mathsf{s}(\mathsf{p}(x) - y))) \quad (31)$$

$$\mathsf{M}(\mathsf{if}(x, y, z)) \to \mathsf{A}(\mathsf{if}(\mathsf{mark}(x), y, z)) \quad (32)$$

$$\begin{aligned} \mathsf{M}(\mathsf{s}(x)) &\to \mathsf{M}(x) \quad (33) & \mathsf{M}(x - y) &\to \mathsf{M}(x) \quad (37) \\ \mathsf{M}(\mathsf{p}(x)) &\to \mathsf{M}(x) \quad (34) & \mathsf{M}(x - y) &\to \mathsf{M}(y) \quad (38) \\ \mathsf{M}(x \leqslant y) &\to \mathsf{M}(x) \quad (35) & \mathsf{M}(\mathsf{if}(x, y, z)) &\to \mathsf{M}(x) \quad (39) \\ \mathsf{M}(x \leqslant y) &\to \mathsf{M}(y) \quad (36) \end{aligned}$$

plus dependency pairs like $\mathsf{S}(f(x)) \to \mathsf{S}(x)$, etc. These latter dependency pairs are only on cycles with themselves and they have no usable rules. Hence the constraints for these cycles of dependency pairs are easily solved by the embedding order.

Dependency pairs (17), (19), (20), (24), (25), (26), and (27) are not on any cycle (this can easily be seen using refinement (B)) and hence we will not consider them further. By combining (31) with (32) and (39) according to refinement (B), we can replace (31) by

$$\mathsf{A}(x - y) \to \mathsf{A}(\mathsf{if}(\mathsf{mark}(x \leqslant y), \mathsf{0}, \mathsf{s}(\mathsf{p}(x) - y))) \tag{40}$$

$$\mathsf{A}(x - y) \to \mathsf{M}(x \leqslant y) \tag{41}$$

Narrowing (40) one step (where we do not have to narrow on $\mathsf{p}(x)$, $\mathsf{p}(x) - y$, or $x \leqslant y$ according to refinement (A)) yields

$$\mathsf{A}(x - y) \to \mathsf{A}(\mathsf{if}(x \leqslant y, \mathsf{0}, \mathsf{s}(\mathsf{p}(x) - y))) \tag{42}$$

$$\mathsf{A}(x - y) \to \mathsf{A}(\mathsf{if}(\mathsf{active}(\mathsf{mark}(x) \leqslant \mathsf{mark}(y)), \mathsf{0}, \mathsf{s}(\mathsf{p}(x) - y))) \tag{43}$$

Moreover, due to refinement (C), in (43) we can replace $\mathsf{mark}(x)$ and $\mathsf{mark}(y)$ by x and y, respectively:

$$\mathsf{A}(x - y) \to \mathsf{A}(\mathsf{if}(\mathsf{active}(x \leqslant y), \mathsf{0}, \mathsf{s}(\mathsf{p}(x) - y))) \tag{44}$$

Now we perform narrowing on (44) (observing refinement (A)) and replace it by the pairs (42) and

$$\mathsf{A}(\mathsf{0} - y) \to \mathsf{A}(\mathsf{if}(\mathsf{true}, \mathsf{0}, \mathsf{s}(\mathsf{p}(\mathsf{0}) - y))) \tag{45}$$

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{0}) \to \mathsf{A}(\mathsf{if}(\mathsf{false}, \mathsf{0}, \mathsf{s}(\mathsf{p}(\mathsf{s}(x)) - \mathsf{0}))) \tag{46}$$

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{s}(y)) \to \mathsf{A}(\mathsf{if}(\mathsf{mark}(x \leqslant y), \mathsf{0}, \mathsf{s}(\mathsf{p}(\mathsf{s}(x)) - \mathsf{s}(y)))) \tag{47}$$

Dependency pairs (42) and (45) are not on a cycle. This is detected by refinement (B), since (42) cannot be combined with any pair and (45) can be combined with (22), but the resulting pair cannot be combined any further. Pair (46) is combined with (23) which yields

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{0}) \to \mathsf{M}(\mathsf{s}(\mathsf{p}(\mathsf{s}(x)) - \mathsf{0})) \tag{48}$$

Pair (47) can be combined with (22) and (23). In order to perform the unification required for the combination, we first have to replace the subterm $\mathsf{mark}(x \leqslant y)$ by a new variable. This yields

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{s}(y)) \to \mathsf{M}(\mathsf{0}) \tag{49}$$

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{s}(y)) \to \mathsf{M}(\mathsf{s}(\mathsf{p}(\mathsf{s}(x)) - \mathsf{s}(y))) \tag{50}$$

Dependency pair (49) is not on a cycle. Both pairs (48) and (50) can be combined with (33) which yields

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{0}) \to \mathsf{M}(\mathsf{p}(\mathsf{s}(x)) - \mathsf{0}) \qquad \mathsf{A}(\mathsf{s}(x) - \mathsf{s}(y)) \to \mathsf{M}(\mathsf{p}(\mathsf{s}(x)) - \mathsf{s}(y))$$

Combining these pairs with (30), (37), and (38) yields

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{0}) \to \mathsf{A}(\mathsf{mark}(\mathsf{p}(\mathsf{s}(x))) - \mathsf{mark}(\mathsf{0})) \tag{51}$$

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{s}(y)) \to \mathsf{A}(\mathsf{mark}(\mathsf{p}(\mathsf{s}(x))) - \mathsf{mark}(\mathsf{s}(y))) \tag{52}$$

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{0}) \to \mathsf{M}(\mathsf{p}(\mathsf{s}(x))) \quad (53) \qquad \mathsf{A}(\mathsf{s}(x) - \mathsf{s}(y)) \to \mathsf{M}(\mathsf{p}(\mathsf{s}(x))) \quad (55)$$

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{0}) \to \mathsf{M}(\mathsf{0}) \quad (54) \qquad \mathsf{A}(\mathsf{s}(x) - \mathsf{s}(y)) \to \mathsf{M}(\mathsf{s}(y)) \quad (56)$$

Dependency pair (54) is not on a cycle. For dependency pair (51) we perform narrowing repeatedly until no further narrowing steps are possible. However, in this process we do not regard narrowing steps which would instantiate variables with terms containing active or mark (due to refinement (A)). Moreover, whenever we encounter a subterm of the form $\mathsf{mark}(x)$, we replace it by x (due to refinement (C)) before continuing the narrowing process. We proceed in an analogous way for dependency pair (52). Thus, these two pairs are transformed into

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{0}) \to \mathsf{A}(x - \mathsf{0}) \quad (57) \qquad \mathsf{A}(\mathsf{s}(x) - \mathsf{s}(y)) \to \mathsf{A}(x - \mathsf{s}(y)) \quad (59)$$

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{0}) \to \mathsf{A}(\mathsf{p}(\mathsf{s}(x)) - \mathsf{0}) \quad (58) \qquad \mathsf{A}(\mathsf{s}(x) - \mathsf{s}(y)) \to \mathsf{A}(\mathsf{p}(\mathsf{s}(x)) - \mathsf{s}(y)) \quad (60)$$

Combining (58) and (60) with (41) yields

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{0}) \to \mathsf{M}(\mathsf{p}(\mathsf{s}(x)) \leqslant \mathsf{0}) \quad (61) \qquad \mathsf{A}(\mathsf{s}(x) - \mathsf{s}(y)) \to \mathsf{M}(\mathsf{p}(\mathsf{s}(x)) \leqslant \mathsf{s}(y)) \quad (62)$$

Pairs (61) and (62) are now combined with (29), (35), and (36), which yields

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{0}) \to \mathsf{A}(\mathsf{mark}(\mathsf{p}(\mathsf{s}(x))) \leqslant \mathsf{mark}(\mathsf{0})) \quad (63)$$

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{0}) \to \mathsf{M}(\mathsf{p}(\mathsf{s}(x))) \quad (53)$$

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{0}) \to \mathsf{M}(\mathsf{0}) \quad (54)$$

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{s}(y)) \to \mathsf{A}(\mathsf{mark}(\mathsf{p}(\mathsf{s}(x))) \leqslant \mathsf{mark}(\mathsf{s}(y))) \quad (64)$$

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{s}(y)) \to \mathsf{M}(\mathsf{p}(\mathsf{s}(x))) \quad (55)$$

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{s}(y)) \to \mathsf{M}(\mathsf{s}(y)) \quad (56)$$

Again, pair (54) is not on a cycle. For (63) and (64) we perform narrowing repeatedly until no further narrowing steps are possible. However, in this process we do not regard narrowing steps which would instantiate variables with terms containing active or mark (due to refinement (A)). Moreover, whenever we encounter a subterm of the form $\mathsf{mark}(x)$, we replace it by x (due to refinement (C)) before continuing the narrowing process. This transforms these two pairs into

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{0}) \to \mathsf{A}(\mathsf{p}(\mathsf{s}(x)) \leqslant \mathsf{0}) \quad (65) \qquad \mathsf{A}(\mathsf{s}(x) - \mathsf{s}(y)) \to \mathsf{A}(\mathsf{p}(\mathsf{s}(x)) \leqslant \mathsf{s}(y)) \quad (67)$$

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{0}) \to \mathsf{A}(x \leqslant \mathsf{0}) \quad (66) \qquad \mathsf{A}(\mathsf{s}(x) - \mathsf{s}(y)) \to \mathsf{A}(x \leqslant \mathsf{s}(y)) \quad (68)$$

Now (65), (66), and (67) are not on a cycle. To summarize, we are left with the following pairs:

$$\mathsf{A}(\mathsf{p}(\mathsf{s}(x))) \to \mathsf{M}(x) \quad (18) \qquad \mathsf{A}(\mathsf{s}(x) - \mathsf{0}) \to \mathsf{A}(x - \mathsf{0}) \quad (57)$$

$$\mathsf{A}(\mathsf{s}(x) \leqslant \mathsf{s}(y)) \to \mathsf{M}(x \leqslant y) \quad (21) \qquad \mathsf{A}(\mathsf{s}(x) - \mathsf{s}(y)) \to \mathsf{A}(x - \mathsf{s}(y)) \quad (59)$$

$$\mathsf{A}(\mathsf{if}(\mathsf{true}, x, y)) \to \mathsf{M}(x) \quad (22) \qquad \mathsf{A}(\mathsf{s}(x) - \mathsf{0}) \to \mathsf{M}(\mathsf{p}(\mathsf{s}(x))) \quad (53)$$

$$\mathsf{A}(\mathsf{if}(\mathsf{false}, x, y)) \to \mathsf{M}(y) \quad (23) \qquad \mathsf{A}(x - y) \to \mathsf{M}(x \leqslant y) \quad (41)$$

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{s}(y)) \to \mathsf{M}(x \leqslant \mathsf{s}(y)) \quad (68) \qquad \mathsf{M}(x - y) \to \mathsf{M}(x) \quad (37)$$

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{s}(y)) \to \mathsf{M}(\mathsf{p}(\mathsf{s}(x))) \quad (55) \qquad \mathsf{M}(x - y) \to \mathsf{M}(y) \quad (38)$$

$$\mathsf{A}(\mathsf{s}(x) - \mathsf{s}(y)) \to \mathsf{M}(\mathsf{s}(y)) \quad (56) \qquad \mathsf{M}(\mathsf{if}(x, y, z)) \to \mathsf{M}(x) \quad (39)$$

$$\mathsf{M}(\mathsf{p}(x)) \to \mathsf{A}(\mathsf{p}(\mathsf{mark}(x))) \qquad (28) \qquad\qquad \mathsf{M}(\mathsf{s}(x)) \to \mathsf{M}(x) \qquad (33)$$

$$\mathsf{M}(x \leqslant y) \to \mathsf{A}(\mathsf{mark}(x) \leqslant \mathsf{mark}(y)) \qquad (29) \qquad\qquad \mathsf{M}(\mathsf{p}(x)) \to \mathsf{M}(x) \qquad (34)$$

$$\mathsf{M}(x - y) \to \mathsf{A}(\mathsf{mark}(x) - \mathsf{mark}(y)) \qquad (30) \qquad\qquad \mathsf{M}(x \leqslant y) \to \mathsf{M}(x) \qquad (35)$$

$$\mathsf{M}(\mathsf{if}(x, y, z)) \to \mathsf{A}(\mathsf{if}(\mathsf{mark}(x), y, z)) \qquad (32) \qquad\qquad \mathsf{M}(x \leqslant y) \to \mathsf{M}(y) \qquad (36)$$

To solve the resulting constraints we use an argument filtering which replaces mark and active by their arguments and RPO with a precedence where "$-$" is greater than both p and "$\leqslant$" and where A and M are equal in the precedence. Then the dependency pairs (28)–(32) are weakly decreasing and all other pairs are strictly decreasing.

Note that $\mathcal{R}$ is an orthogonal constructor system where all argument positions of constructors are active. Moreover, in the above dependency pairs of $\mathcal{R}_\mu^3$, defined symbols of $\mathcal{R}$ occur only at position 1 in right-hand sides. Hence, refinement (D) is applicable which implies that the active-rules are not usable. As a consequence, by the above argument filtering, the left and right-hand sides of all usable rules are made equal. In other words, the constraints resulting from the usable rules are fulfilled. Hence, the transformed system is innermost terminating and thus, the original CSRS is also innermost terminating. Since the CSRS is orthogonal, this also implies its termination.

This example demonstrates that our results are also helpful for termination proofs of such CSRSs, because they imply that it is sufficient to prove *innermost* termination of the transformed system. In general, proving innermost termination is significantly easier than proving termination [1]. Indeed, in our proof we made use of several refinements of the dependency pair approach which can only be used for innermost termination proofs:

- Refinements (A)–(E) only work for innermost termination.
- The technique of narrowing dependency pairs (for non-right-linear systems like $\mathcal{R}_\mu^3$) can only be used for innermost termination.
- The technique of usable rules only works for innermost termination (this is also important when handling the $\mathsf{S}(f(x)) \to \mathsf{S}(x)$ dependency pairs which have no usable rules).

A.3 Refinements to the Dependency Pair Approach

In this section we comment on the correctness of the refinements (A)–(E) that were used in the preceding examples.

A.3.1 Refinement (A)

In innermost chains one only regards instantiations of dependency pairs where the left-hand side is a normal form. Since there is a symbol $f \in \mathcal{F}$ above every variable in the left-hand side of every A or M-dependency pair, it follows that the variables in these pairs cannot be be instantiated by terms containing active or mark. Hence, in A or M-dependency pairs, no narrowing is needed which would instantiate variables with terms containing active or mark.

A.3.2 Refinement (B)

Refinement (B) is a special case of the following refinement, which can be used for dependency pairs in general.

Theorem 35 (combining dependency pairs). *Let $\mathcal{R}$ be a TRS, let $\mathcal{P}$ be a set of pairs of terms such that $\mathcal{V}\mathrm{ar}(v) \subseteq \mathcal{V}\mathrm{ar}(u)$ for all $u \to v \in \mathcal{P}$, and let $s \to t \in \mathcal{P}$. Let $t = t'\tau$ with $\mathcal{D}\mathrm{om}(\tau) = \mathcal{V}\mathrm{ar}(t') \setminus \mathcal{V}\mathrm{ar}(t)$ such that for all σ with $s\sigma$ a normal form and $\mathcal{D}\mathrm{om}(\sigma) \cap \mathcal{D}\mathrm{om}(\tau) = \varnothing$, any normal form of $t\sigma$ has the form $t'(\sigma \cup \tau')$ for some τ' with $\mathcal{D}\mathrm{om}(\tau') \subseteq \mathcal{D}\mathrm{om}(\tau)$. Let*

$$\mathcal{P}' = \mathcal{P} \setminus \{s \to t\} \cup \{s\mu \to v\mu \mid u \to v \in \mathcal{P} \text{ and } \mu = \mathrm{mgu}(t', u)\}$$

If there exists no infinite innermost $\mathcal{R}$-chain of pairs from $\mathcal{P}'$, then there exists no infinite innermost $\mathcal{R}$-chain of pairs from $\mathcal{P}$.

Proof. If

$$\cdots,\ s \to t,\ u \to v,\ \cdots$$

is an innermost chain of pairs from $\mathcal{P}$, then there exists a substitution σ such that $s\sigma$ and $u\sigma$ are normal forms and such that $t\sigma \xrightarrow{\mathrm{i}}{}^{*}_{\mathcal{R}} u\sigma$. Since τ' only operates on the new variables in t' we have $u\sigma = t'(\sigma \cup \tau') = u(\sigma \cup \tau')$. Hence, $\sigma \cup \tau'$ is a unifier of t' and u. Let μ be the mgu of these two terms. So there exists a substitution ρ such that $\sigma \cup \tau' = \mu\rho$. Hence, the two dependency pairs $s \to t$ and $u \to v$ in the innermost chain can be replaced by the new pair $s\mu \to v\mu$ where instead of the instantiation σ one now has to use the instantiation ρ. ☐

Recall that the variables in the A and M-dependency pairs cannot be instantiated by terms containing active or mark. Thus, the symbols from $\mathcal{F}$ occurring in right-hand sides of dependency pairs can be treated like constructors when using the technique of combining dependency pairs. In other words, all dependency pairs $s \to t$ with $\mathrm{root}(s) \in \{\mathsf{A}, \mathsf{M}\}$ and no active and mark-symbols occurring in t have the property required in Theorem 35, i.e., for all σ where $s\sigma$ is a normal form, $t\sigma$ is a normal form, too.

Due to the form of $\mathcal{R}^3_\mu$, for arbitrary terms t the following holds. Let $t = C[t_1, \ldots, t_k]$ where the context C does not contain mark and active-symbols, and where the root symbol of the terms t_i is mark or active. For substitutions σ which do not introduce mark or active-symbols, $t\sigma$ has the form $C\sigma[t_1\sigma, \ldots, t_k\sigma]$ and again, $C\sigma$ does not contain mark and active-symbols. Note that $t\sigma$ can only rewrite in $\mathcal{R}^3_\mu$ to terms of the form $C\sigma[u_1, \ldots, u_k]$. Hence, if we replace all mark and active-subterms in t by pairwise different fresh variables then the resulting term $\textsc{cap}'(t)$ satisfies the requirements on the term t' in Theorem 35.

A.3.3 Refinement (C)

First note that if $\mathcal{R}_\mu^3$ is not innermost terminating, then there also exists an infinite innermost $\mathcal{R}_\mu^3$-reduction with terms from T where T is defined as in Definition 19. (To see this, note that if $\mathcal{R}_\mu^3$ is not innermost terminating then $(\mathcal{R}, \mu)$ is not innermost terminating as a consequence of the completeness of Θ_3. From the proof of Theorem 18 we then infer the existence of an infinite innermost reduction starting from a term of the form $\mathsf{mark}(s)\downarrow_{\mathcal{M}}$. Obviously, $\mathsf{mark}(s)\downarrow_{\mathcal{M}} \in T$.) Further note that according to the proof of Lemma 20, the set T is closed under innermost $\mathcal{R}_\mu^3$-reduction.

Now we show that without loss of generality we can assume that in the left-hand side $\mathsf{A}(C[x])$ of every A-dependency pair with x on an active position of $C[x]$, x can only be instantiated with terms $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ such that for all active positions π in s, $s|_\pi$ is not an $\mathcal{R}$-redex. Every infinite innermost $\mathcal{R}_\mu^3$-reduction corresponds to an infinite innermost chain of dependency pairs. As explained above, we can restrict ourselves to reductions between terms of T. Then an instantiation of a dependency pair $\mathsf{A}(C[x]) \to \ldots$ with a substitution σ can only occur in this innermost chain if there is a term $t = t[\mathsf{active}(C\sigma[x\sigma])]_{\pi'} \in T$ in the infinite innermost $\mathcal{R}_\mu^3$-reduction. According to the definition of T, the position π' of the displayed occurrence of active in t is active. Because the position of x is active in $C[x]$, it is also active in t. Let $s = x\sigma$. Due to the form of the dependency pairs, $C[x]$ contains at least one symbol of $\mathcal{F}$ above x. Moreover, in innermost chains, the variables of A or M-dependency pairs cannot be instantiated by terms containing active or mark, cf. the argumentation for refinements (A) and (B). From these two observations we infer that the active positions π of s are not activated. Hence, by the definition of T, $s|_\pi$ is not an $\mathcal{R}$-redex. Thus, we can indeed assume that in dependency pairs $\mathsf{A}(C[x]) \to \ldots$ with x on an active position of $C[x]$, x is only instantiated with terms $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ without $\mathcal{R}$-redexes on active positions.

Note that for such terms s, the normal form of $\mathsf{mark}(s)$ (reachable by innermost $\mathcal{R}_\mu^3$-reduction) is $\mathsf{mark}(s)$ or $\mathsf{active}(s)$. To see this, we consider two cases. If $s \in \mathcal{V}$ then $\mathsf{mark}(s)$ is a normal form. Otherwise, by Lemma 17 the normal form of $\mathsf{mark}(s)$ is $\mathsf{active}(s)$ since any innermost $\mathcal{R}_\mu^3$-reduction would first reduce $\mathsf{mark}(s)$ to $\mathsf{mark}(s)\downarrow_{\mathcal{M}}$. Hence, any instantiation of a dependency pair

$$\mathsf{A}(C[x]) \to C'[f(\ldots, \mathsf{mark}(x), \ldots)]$$

will only lead to a right-hand side that is reduced to $C'[f(\ldots, \mathsf{active}(s), \ldots)]$ or $C'[f(\ldots, \mathsf{mark}(s), \ldots)]$ and then to $C'[f(\ldots, s, \ldots)]$. Hence, one can immediately replace the right-hand side by $C'[f(\ldots, x, \ldots)]$.

A.3.4 Refinement (D)

We have the following theorem.

Theorem 36. *Let $(\mathcal{R}, \mu)$ be an orthogonal CSRS which is a constructor system with $\mu(c) = \{1, \ldots, \text{arity}(c)\}$ for all constructors c. If $(\mathcal{R}, \mu)$ is not innermost terminating then there exists a term without defined symbols below the root which starts an infinite innermost reduction.*

Proof. For an innermost terminating term u we denote by $\vartheta(u)$ the result of replacing in the unique μ-normal form of u all subterms with defined root symbol by a (distinguished) variable. We call a term that is obtained from a term s by replacing some occurrences of some innermost terminating subterms u by $\vartheta(u)$ a *normal form variant* of s.

Let $s \xrightarrow{\mathsf{i}}_\mu t$ and let s' be a normal form variant of s. We claim that there exists a normal form variant t' of t such that $s' \xrightarrow{\mathsf{i}}{}^{=}_{\mu} t'$. To prove the claim we distinguish two cases.

1. We first regard the case where in the step from s to t a rule $l \to r$ is applied to a redex not inside one of the replaced subterms. Thus we have $s = C[l\sigma]_\pi$ and $t = C[r\sigma]_\pi$ such that no subterm on or above position π is replaced in s'. Since constructors have only active argument positions, replacing a term u by $\vartheta(u)$ is the same as replacing all subterms u' of u with defined root symbol by $\vartheta(u')$. Hence, without loss of generality we can assume that below π one only replaces subterms with defined root symbol. Since $\mathcal{R}$ is a constructor system, all subterms with defined root symbol are introduced by the substitution σ. Since $\mathcal{R}$ is orthogonal, the replacement of such subterms u' by $\vartheta(u')$ corresponds to the use of a modified substitution σ'. So $s' = C'[l\sigma']_\pi$ for a suitable substitution σ'. The resulting term $t' = C'[r\sigma']_\pi$ is easily seen to be a normal form variant of t and the step from s' to t' is innermost.
2. If the step from s to t takes place inside a replaced subterm u of s then we have $s = C[u]$ and $t = C[v]$ with $u \xrightarrow{\mathsf{i}}_\mu v$. In the normal form variant s', the term u has been replaced by $\vartheta(u)$. We have $\vartheta(u) = \vartheta(v)$ since u and v have the same μ-normal form. Hence, $s' = C'[\vartheta(u)] = C'[\vartheta(v)]$ is a normal form variant of t.

Let s be a minimal term with an infinite innermost reduction, i.e., all proper subterms are innermost terminating. From the preceding discussion we infer that the normal form variant s' of s obtained by replacing *all* proper subterms u of s by $\vartheta(u)$ again has an infinite innermost reduction; note that the second alternative above happens only finitely many times as the replaced subterms are innermost terminating. Since s' has no defined symbols below the root, this proves the theorem. $\square$

One should remark that the above theorem does not hold if constructors have inactive argument positions. As a counterexample consider the CSRS consisting of the two rules

$$\mathsf{f}(\mathsf{c}(x)) \to \mathsf{f}(x) \qquad\qquad \mathsf{b} \to \mathsf{c}(\mathsf{b})$$

with $\mu(\mathsf{f}) = \{1\}$ and $\mu(\mathsf{c}) = \varnothing$. The term $\mathsf{f}(\mathsf{c}(\mathsf{b}))$ starts an infinite innermost context-sensitive reduction, but all terms without defined symbols below the root are innermost terminating.

Furthermore, the orthogonality requirement cannot be weakened to non-overlappingness, as can be seen from the CSRS

$$\mathsf{f}(x) \to \mathsf{g}(x, \mathsf{i}(\mathsf{h}(\mathsf{c}))) \qquad \mathsf{g}(x, x) \to \mathsf{f}(\mathsf{h}(\mathsf{b})) \qquad \mathsf{h}(\mathsf{b}) \to \mathsf{i}(\mathsf{h}(\mathsf{c}))$$

with the argument positions of all function symbols active. The term $\mathsf{f}(\mathsf{h}(\mathsf{b}))$ starts an infinite innermost reduction, but all terms without defined symbols below the root are innermost terminating.

According to Theorem 36, if $(\mathcal{R}, \mu)$ is not innermost terminating, then there exists a term $f(t_1, \ldots, t_n)$ with an infinite innermost μ-reduction such that the terms $t_1, \ldots, t_n$ contain no $\mathcal{R}$-defined function symbols. Then $\mathsf{mark}(f(t_1, \ldots, t_n))$ has an infinite innermost $\mathcal{R}^3_\mu$-reduction. The reason is that since $t_1, \ldots, t_n$ contain no $\mathcal{R}$-redexes, any innermost $\mathcal{R}^3_\mu$-reduction would first reduce $\mathsf{mark}(f(t_1, \ldots, t_n))$ to $\mathsf{mark}(f(t_1, \ldots, t_n))\!\downarrow_{\mathcal{M}}$. Then the claim follows from the proof of Theorem 18.

The term $\mathsf{mark}(f(t_1, \ldots, t_n))$ is obviously a *minimal* term with an infinite innermost $\mathcal{R}^3_\mu$-reduction, i.e., all its subterms are innermost terminating with respect to $\mathcal{R}^3_\mu$. From the soundness proof of the dependency pairs technique [1, Theorems 31 and 6] one can see that every minimal non-innermost terminating term $f_1(\boldsymbol{u_1})$ gives rise to an infinite innermost chain of dependency pairs

$$F_1(\boldsymbol{v_1}) \to F_2(\boldsymbol{u_2}),\ F_2(\boldsymbol{v_2}) \to F_3(\boldsymbol{u_3}),\ \cdots$$

such that every $f_i(\boldsymbol{v_i}) \to r_i$ is a rewrite rule, $f_{i+1}(\boldsymbol{u_{i+1}})$ is a subterm of r_i, there are substitutions σ_i such that $F_{i+1}(\boldsymbol{u_{i+1}})\sigma_i \xrightarrow{\mathsf{i}}{}^*_3 F_{i+1}(\boldsymbol{v_{i+1}})\sigma_{i+1}$, and every $F_{i+1}(\boldsymbol{v_{i+1}})\sigma_{i+1}$ is a normal form. Moreover, we have $F_1(\boldsymbol{u_1}) \xrightarrow{\mathsf{i}}{}^*_3 F_1(\boldsymbol{v_1})\sigma_1$. Hence, in our setting there is an infinite innermost chain starting with a dependency pair whose left-hand side is $\mathsf{M}(f(\ldots))$ and σ_1 instantiates the variables of this dependency pair by terms without $\mathcal{R}$-defined symbols.

By assumption no right-hand side $F_i(\boldsymbol{u_i})$ of a dependency pair contains an $\mathcal{R}$-defined symbol strictly below position 1. Hence, if $\mathcal{R}$-defined symbols only occur at position 1 in an instantiated left-hand side $F_i(\boldsymbol{v_i})\sigma_i$ of a dependency pair, then this also holds for the instantiated right-hand side $F_i(\boldsymbol{u_{i+1}})\sigma_i$ of the pair. Hence, in the instantiated dependency pairs, mark is only applied to terms which contain no $\mathcal{R}$-defined function symbols. The terms resulting from these reductions again contain no $\mathcal{R}$-defined symbols. It follows that the only usable rules are rules of the form $\mathsf{mark}(f(\ldots)) \to \mathsf{active}(f(\ldots))$ for constructors f and rules $f(\ldots, g(x), \ldots) \to f(\ldots, x, \ldots)$ for $f \in \mathcal{F}$ and $g \in \{\mathsf{mark}, \mathsf{active}\}$.

A.3.5 Refinement (E)

The following theorem holds for arbitrary TRSs.

Theorem 37. *Let $\mathcal{R}$ be a TRS and let $\mathcal{P}$ be a set of pairs of terms. Let $s \to t \in \mathcal{P}$, let $t|_\pi$ be a reducible ground term, and let $u_1, \ldots, u_n$ be the terms reachable from $t|_\pi$ in one innermost rewrite step. Let $\mathcal{P}'$ result from $\mathcal{P}$ by replacing $s \to t$ with $s \to t[u_1]_\pi$, $\ldots$, $s \to t[u_n]_\pi$. If there exists no infinite innermost chain of pairs from $\mathcal{P}'$, then there exists no infinite innermost chain from $\mathcal{P}$ either.*

Proof. Let

$$\cdots,\ s \to t,\ u \to v,\ \cdots$$

be an innermost chain of pairs from $\mathcal{P}$. Then there must be a substitution σ with $t\sigma \xrightarrow{\mathrm{i}}{}^*_{\mathcal{R}} u\sigma$ where $u\sigma$ is a normal form. Since $t\sigma|_\pi = t|_\pi$ is reducible, there is at least one rewrite step in this reduction. Since the reduction is innermost, $t|_\pi$ must be normalized before reduction steps are applied to positions above π in $t\sigma$. Obviously, it does not matter in which order reduction steps are performed on pairwise disjoint positions. Hence, we can assume that in the reduction $t\sigma \xrightarrow{\mathrm{i}}{}^*_{\mathcal{R}} u\sigma$ one first normalizes $t|_\pi$. So there exists a term u_i such that $t\sigma \xrightarrow{\mathrm{i}}_{\mathcal{R}} t[u_i]_\pi\sigma \xrightarrow{\mathrm{i}}{}^*_{\mathcal{R}} u\sigma$. Hence, we can replace the dependency pair $s \to t$ by $s \to t[u_i]_\pi$ in the above innermost chain. □

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