Analysis of mean square estimator errors of basic frequency for periodically non-stationary random processes*

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Abstract

Functional for estimation of the basic frequency for the mean and covariance functions of periodically non-stationary random processes (PNRPs), grounded on the reduced LS functional, analyzed. It is obtained that for the case of a Gaussian PNRP, the maximum points of the proposed functional are unbiased and consistent estimators of the basic frequency if the covariance function decays with time lag. Such analysis is provided using solutions of the nonlinear equations with appropriate conditions for maximum existence. The solutions are obtained using the small parameter method.

Keywords

periodically non-stationary random process, basic frequency estimator, quasi-optimal functional, consistency

1. Introduction

Periodically non-stationary random processes (PNRPs) are the stochastic models of hidden periodicities. Such models can be applied to describe the irregularity and recurrence is time series, which explains different natural processes. When representing a PNRP in the form of a series [1, 2]:

$$\xi(t)=\sum_{k\in Z}\xi_k(t)e^{ik\omega_0 t}=\xi_0(t)+\sum_{k\in N}\left[\xi_k^c(t)\cos k\omega_0 t+\xi_k^s(t)\sin k\omega_0 t\right], \tag{1}$$

where $\omega_0=\frac{2\pi}{P}$ is the basic frequency, P is the period, $\xi_0(t)$ and $\xi_k(t)=\frac{1}{2}\left[\xi_k^c(t)-i\xi_k^s(t)\right]$ are jointly stationary random processes.

The moment functions of processes $\xi_k(t)$ determines the periodic changes in the mean $m(t)=E\xi(t)$ and covariance $R(t,\tau)=E\overset{\circ}{\xi}(t)\overset{\circ}{\xi}(t+\tau)$, functions ($\overset{\circ}{\xi}(t)=\xi(t)-m(t)$), which can be represented in the form of a Fourier series in a following way:

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$$m(t)=\sum_{k\in Z} m_k e^{ik\omega_0 t}=m_0+\sum_{k\in N}\left[m_k^c\cos k\omega_0 t+m_k^s\sin k\omega_0 t\right],\tag{2}$$

$$R(t,\tau)=\sum_{k\in Z} R_k(\tau)e^{ik\omega_0 t}=R_0(\tau)+\sum_{k\in N}\left[R_k^c(\tau)\cos k\omega_0 t+R_k^s(\tau)\sin k\omega_0 t\right],\tag{3}$$

where $m_k=\frac{1}{2}\left(m_k^c-im_k^s\right)$ and $R_k(\tau)=\frac{1}{2}\left[R_k^c(\tau)-iR_k^s(\tau)\right]$. The Fourier coefficients of the series in (2) and (3) are defined by the modulating processes $\xi_k(t)$, and expressed as $m_k=E\xi_k(t)$ and

$$R_k(\tau)=\sum_{l\in Z} R_{l-k,l}(\tau)e^{il\omega_0\tau},\tag{4}$$

Discovering the hidden periodicities using the model of signal as PNRP has been considered in a number of works [3–17]. The statistical properties of the estimator of the non-stationarity period were not analyzed in these investigations, but were briefly characterized in [18, 19].

As shown in [2], the biases and the variances of the LS estimators for the mean and covariance functions and the component estimators formed on the basis of their cyclic statistics quickly converge as the realization length draws.

2. Mean square functional

It was proved in [20, 21] that the LS estimation of the basic frequency for the PNRP mean function can be reduced to finding the maximum point of the following squared functional:

$$\hat{F}_1(\omega)=\frac{1}{2T}\int_{-T}^{T}\hat{m}^2(\omega,t)\,dt,\tag{5}$$

where

$$\hat{m}(\omega,t)=\hat{m}_0(\omega)+\sum_{k=1}^{L}\left[\hat{m}_k^c(\omega)\cos k\omega t+\hat{m}_k^s(\omega)\sin k\omega t\right],\tag{6}$$

and L in (6) is the number of chosen harmonics, $2T$ is the realization length.

We substitute into the functional in (5) the component estimator for the mean function [1, 2] rather than its LS estimator.

From the time averaging of (5), we have:

$$\begin{aligned}\hat{F}_1(\omega)&=\hat{m}_0^2+\frac{1}{2}\sum_{k=1}^{L}\left[\left[\hat{m}_k^c(\omega)\right]^2+\left[\hat{m}_k^s(\omega)\right]^2\right]+\\&+\sum_{\substack{k,l=1\\k\neq l}}^{L}\left[\hat{m}_k^c(\omega)\hat{m}_l^s(\omega)+\hat{m}_k^s(\omega)\hat{m}_l^c(\omega)\right]J_0\left[(k-l)\omega,T\right]+\\&+\sum_{k,l=1}^{L}\left[\hat{m}_k^c(\omega)\hat{m}_l^c(\omega)-\hat{m}_k^s(\omega)\hat{m}_l^s(\omega)\right]J_0\left[(k+l)\omega,T\right],\end{aligned}\tag{9}$$

where $J_0(\omega,T)=sin\omega T/\omega T$. Terms, that depend on the functions $J_0(\omega,T)$, vanish as T increases. They are the higher order smallness in comparison to first two terms in equation (9). So, we have:

$$\hat{Q}_1(\omega)=\frac{1}{2}\sum_{k=1}^{L}\left[\left[\hat{m}_k^c(\omega)\right]^2+\left[\hat{m}_k^s(\omega)\right]^2\right] \tag{10}$$

Let us consider the properties of the maximum point of the functional in (10). This point can be found as a solution to the following nonlinear equation:

$$\sum_{k=1}^{L}\left[\hat{m}_k^c(\omega)\frac{d\hat{m}_k^c(\omega)}{d\omega}+\hat{m}_k^s(\omega)\frac{d\hat{m}_k^s(\omega)}{d\omega}\right]=0 \tag{11}$$

When to introduce in (10) the normalized deterministic and fluctuation parts as follows:

$$C_k^{(n)}(\omega)=\frac{C_k(\omega)}{\left[P_t^{(d)}\right]^{\frac{1}{2}}},\quad S_k^{(n)}(\omega)=\frac{S_k(\omega)}{\left[P_t^{(d)}\right]^{\frac{1}{2}}},$$

$$M_k^{(n)}(\omega)=\frac{M_k(\omega)}{\left[D_t\left[\hat{m}(t)\right]\right]^{\frac{1}{2}}},\quad N_k^{(n)}(\omega)=\frac{N_k(\omega)}{\left[D_t\left[\hat{m}(t)\right]\right]^{\frac{1}{2}}}.$$

And denote:

$$c_k^{(l)}=\left[\frac{d^l C_k^{(n)}(\omega)}{d\omega^l}\right]_{\omega=\omega_0},\quad s_k^{(l)}=\left[\frac{d^l S_k^{(n)}(\omega)}{d\omega^l}\right]_{\omega=\omega_0},$$

$$m_k^{(l)}=\left[\frac{d^l M_k^{(n)}(\omega)}{d\omega^l}\right]_{\omega=\omega_0},\quad n_k^{(l)}=\left[\frac{d^l N_k^{(n)}(\omega)}{d\omega^l}\right]_{\omega=\omega_0},$$

we can decompose the left-hand side of the equation (11) into a Taylor series in the neighborhood of the point $\omega=\omega_0$. Then for a first approximation, we will have:

$$\Delta\omega=\frac{1}{p(T)}\sum_{k=1}^{L}\left(c_k^{(0)}c_k^{(1)}+s_k^{(0)}s_k^{(1)}\right), \tag{12}$$

$$\omega_1=\frac{1}{p(T)}\sum_{k=1}^{L}\left[\begin{array}{l}c_k^{(0)}m_k^{(1)}+s_k^{(0)}n_k^{(1)}+s_k^{(1)}n_k^{(0)}+c_k^{(1)}m_k^{(0)}+\\+\Delta\omega\left(2c_k^{(1)}m_k^{(1)}+c_k^{(0)}m_k^{(2)}+c_k^{(2)}m_k^{(0)}+s_k^{(0)}n_k^{(2)}+s_k^{(2)}n_k^{(0)}\right)\end{array}\right], \tag{13}$$

where

$$p(T)=\sum_{k=1}^{L}\left[\left(c_k^{(1)}\right)^2+\left(s_k^{(1)}\right)^2+c_k^{(0)}c_k^{(2)}+s_k^{(0)}s_k^{(2)}\right]. \tag{14}$$

Now, if $R(t,\tau)$ tends to zero $\forall t\in[0,P]$ as τ increases, i.e: $\lim_{|\tau|\to\infty}R(t,\tau)=0$ or $\lim_{|\tau|\to\infty}R_k(\tau)=0, k=\overline{1,2L}$, the ω-value that produces the maximum of the functional in (10) for a Gaussian PNRP is an asymptotically unbiased and consistent estimator for the basic frequency of the mean function, and to a first approximation, its bias $\varepsilon\left[\hat{\omega}_0\right]=E\hat{\omega}_0-\omega_0$ and variance $D\left[\hat{\omega}_0\right]=E\hat{\omega}_0^2-\left[E\hat{\omega}_0\right]^2$ are defined by the formulae:

$$\varepsilon\left[\hat{\omega}_0\right]=\frac{3}{T^2\sum\limits_{k=1}^{L}k^2\left[\left(m_k^c\right)^2+\left(m_k^s\right)^2\right]}\sum_{k=1}^{L}k\left[m_k^c\sum_{l=1}^{L}m_l^c\left[J_1\left[\left(l-k\right)\omega_0,T\right]+J_1\left[\left(l+k\right)\omega_0,T\right]\right]-\right.$$
$$\left.-m_k^s\sum_{l=1}^{L}m_l^s\left[J_1\left[\left(l-k\right)\omega_0,T\right]+J_1\left[\left(l+k\right)\omega_0,T\right]\right]\right]+\mathrm{o}\left(T^{-2}\right) \tag{15}$$

$$D\left[\hat{\omega}_0\right]=\frac{3}{T^3\left[\sum\limits_{k=1}^{L}k^2\left[\left(m_k^c\right)^2+\left(m_k^s\right)^2\right]\right]^2}\times$$
$$\times\sum_{l,k=1}^{L}kl\int_0^{2T}\left[m_k^c m_l^c\left[R_{k+l}^s\left(u\right)\left(\sin k\omega_0u+\sin l\omega_0u\right)+R_{k-l}^s\left(u\right)\left(\sin k\omega_0u-\sin l\omega_0u\right)+\right.\right.$$
$$+\left[R_{k-l}^c\left(u\right)-R_{k+l}^c\left(u\right)\right]\left(\cos k\omega_0u+\cos l\omega_0u\right)\Big]+m_k^s m_l^s\left[R_{k-l}^s\left(u\right)\left(\sin k\omega_0u-\sin l\omega_0u\right)-\right.$$
$$-R_{k+l}^s\left(u\right)\left(\sin k\omega_0u+\sin l\omega_0u\right)+\left[R_{k+l}^c\left(u\right)+R_{k-l}^c\left(u\right)\right]\left(\cos k\omega_0u+\cos l\omega_0u\right)\Big]-$$
$$-2m_k^c m_l^s\left[R_{l+k}^c\left(u\right)\left(\sin k\omega_0u+\sin l\omega_0u\right)+R_{k-l}^s\left(u\right)\left(\sin k\omega_0u-\sin l\omega_0u\right)+\right.$$
$$\left.\left.+\left[R_{k+l}^s\left(u\right)+R_{l-s}^s\left(u\right)\right]\left(\cos k\omega_0u+\cos l\omega_0u\right)\right]\right]du+\mathrm{o}\left(T^{-3}\right), \tag{16}$$

where $J_1\left[\omega,T\right]=\dfrac{\sin\omega T}{\omega^2T}-\dfrac{\cos\omega T}{\omega}\quad\forall\,\omega\neq0$.

3. Conclusion

It was shown that LS estimation of the basic frequency for mean and covariance functions of the PNRP can be reduced to searching for the maximum points of the simplified functional, which are defined by the sums of the powers of the harmonics of the test frequency and its multiples. The values of the maximum points of functional are close to the time-averaged powers of the time changes of related moment function. It was proved that the estimator, obtained with such functional for a Gaussian PNRP is asymptotically unbiased and consistent if the variance function tends to zero with time lag increasing. Formulae for the estimator variances were derived to a first approximation.

Declaration on Generative AI

The authors have not employed any Generative AI tools.

References

[1] S.M. Kay, Modern Spectral Estimation: Theory and Application, Prentice Hall, New Jersey, 1988.

[2] I. Javorskyj, R. Yuzefovych, O. Dzeryn, M. Varyvoda, Analysis of Deterministic Components of Biperiodically Correlated Random Signals, 10th International Conference on Advanced Computer Information Technologies, ACIT 2020, 65–68.

[3] I. Javorskyj, R. Yuzefovych, O. Lychak, R. Slyepko, P. Semenov, Detection of distributed and localized faults in rotating machines using periodically non-stationary covariance analysis of vibrations, Measurement Science and Technology. 34 (2023) 065102.
[4] H.L. Hurd, N.L. Gerr, Graphical methods for determining the presence of periodic correlation, J. Time Ser. Anal. 12 (1991) 337-350.
[5] H. L. Hurd, A. Miamee, Periodically Correlated Random Sequences: Spectral Theory and Practice. Wiley, New York, 2007. doi: 10.1002/9780470182833.
[6] A.V. Dandowate, G. B. Giannakis, Statistical tests for presence of periodic correlation, IEEE Trans. Signal Process. 42 (1994) 2355-2369.
[7] G. Jeuny, W. Gardner, Search-efficient method of detection of cyclostationary signals, IEEE Trans. Signal Process. 44 (1996) 2355-2369.
[8] P. Ciblat, P. Loubaton, E. Serpedin, G.B. Giannakis, Performance analysis of blind career frequency offset estimators for noncircular transmissions through frequency-selective channels, IEEE Trans. Signal Process. 50(1) (2002) 130-140.
[9] P. Ciblat, P. Loubaton, E. Serpedin, G.B. Giannakis, Asymptotic analysis of blind cyclic correlation-based symbol-rate estimators, IEEE Trans. Inf. Theory 48(7) (2002) 1922-1934.
[10] J. Leskow, Cyclostationary and resampling for vibroacoustic signals, Acta Thus. Pol. 121 (1A) (2012) 160-163.
[11] W. Cioch, O. Knapik, J. Leskow, Finding a frequency signature for a cyclostationary signal with applications to wheel bearing diagnostics, Mech. Syst. Signal Process 38 (2013) 55-64.
[12] S. Luan, T. Qju, Y. Zhu, L. Yu, Cyclic correntropy and its spectrum in frequency estimation in a presence of impulsive noise, Signal Process. 120 (2016) 503-508.
[13] V. Kyryliv, Ya. Kyryliv, N. Sas, Formation of surface ultrafine grain structure and their physical and mechanical characteristics using vibration-centrifugal hardening, Advances in Materials Science and Engineering, 2018, Article number 3152170. doi: 10.1155/2018/3152170.
[14] I. Dolinska, Evaluation of the residual service life of a disk of the rotor of steam turbine with regard for the number of shutdowns of the equipment, Mater. Sci. 53(5) (2018) 637–644. doi: 10.1007/s11003-018-0118-y.
[15] A. M. Syrotyuk, A. V. Babii, R. A. Barna, R. L. Leshchak, P. O. Marushchak, Corrosion-Fatigue crack-growth resistance of steel of the frame of a sprayer boom, Mater. Sci. 56(4) (2021) 466–471. doi: 10.1007/s11003-021-00452-2.
[16] Z. K. Zhu, Z. H. Feng, F. R. Kong, Cyclostationarity analysis for gearbox condition monitoring: Approaches and effectiveness, Mech. Syst. Signal Process., 19(3) (2005) 467–482. doi: 10.1016/j.ymssp.2004.02.007.
[17] D. Ho, R. B. Randall, Optimization of bearing diagnostic techniques using simulated and actual bearing fault signals, Mech. Syst. Signal Process. 14(5) (2000) 763–788. doi: 10.1006/mssp.2000.1304.
[18] S. Tyagi, S. K. Panigrahi, An improved envelope detection method using particle swarm optimisation for rolling element bearing fault diagnosis, J. Comput. Des. Eng. 4 (2017) 305–317. doi: 10.1016/j.jcde.2017.05.002.
[19] Y. Liu, T. Qju, H. Sheng, Time-difference-of-arrival estimation algorithms for cyclostationary signals in impulsive noise, Signal Process. 92 (2012) 2238-2247
[20] I. Javorskyj, R. Yuzefovych, I. Matsko, Z. Zakrzewski, The least square estimation of the basic frequency for periodically non-stationary random signals, Digit. Signal Process. 122 (2022) 103333.
[21] I. Javorskyj, R. Yuzefovych, O. Dzeryn, P. Semenov, Properties of LSM-estimator of correlation function of biperiodically correlated random processes, J. Aut. Inf. Sci. 52 (2020) 44-50.