Improvement of the mathematical models reduction principle by group elements deletion*

Yaroslav Matviychuk [1*,†], Nataliya Melnykova[1,†]

[1] *Department of Systems of Artificial Intelligence, Lviv Polytechnic National University, S. Bandera, str. 12, Lviv, 79013, Ukraine*

Abstract
The classic principle of reduction suggests removing unnecessary elements of the model one at a time [2]. The reduction process can be significantly accelerated by group removal of elements. The article substantiates such removal and shows its effectiveness on the example of the identification of the Lorenz ODE.

Keywords
Mathematical model, reduction, identification, incorrectness, ODE

1.Introduction

Mathematical modeling is one of the foundations of scientific knowledge. The identification of the mathematical model is reversed problem, and therefore incorrect [1]. Identification errors can occur from redundant model elements [2].

The classic principle of reduction [3], [4] suggests removing unnecessary elements of the model one at a time. If the model is complex, with a large number of elements, then such reduction may take too much time.

In this paper we propose the improvement of the reduction principle to reduce reduction time. Using the Lorenz atractor equations as an example, it is shown how to do group deletion of model elements. Group deletion significantly speeds up the model reduction process.

1.1. Reduction principle

Assume there exists a precise mathematical model for a simulated object, characterized by a known vector of parameters $\mathbf{p} = (\mathbf{p}_1,\ldots,\mathbf{p}_n)$. These parameters are determined through a specific identification method. A parameter value of zero implies the absence of the corresponding element, and the identification problem—finding the vector **p** — is solved based on the continuous dependence of the solution on the initial conditions within a certain neighborhood.

Now, let the parameter vector be expanded to include additional components: $\mathbf{p} = (\mathbf{p}_1,\ldots,\mathbf{p}_n,\ \mathbf{p}_{n+1},\ldots,\mathbf{p}_m)$. In this extended scenario, the identification method will assign values near zero to the superfluous parameters $\mathbf{p}_{n+1},\ldots,\mathbf{p}_m$, with deviations only within the bounds of computational precision:

$$p_i \approx 0;\ \ i = n+1,\ldots,m. \qquad (1)$$

*SmartIndustry 2025: 2nd International Conference on Smart Automation & Robotics for Future Industry, April 03-05, 2025, Lviv, Ukraine

1* Corresponding author.
† These authors contributed equally.
matv@ua.fm (Y. Matviychuk); nataliia.i.melnykova@lpnu.ua (N. Melnykova
0000-0002-5570-182X (Y. Matviychuk); 0000-0002-2114-3436 (N. Melnykova);

But this property can not detect unnecessary parameters, because some of the needed parameters may be close to zero. Next, we apply slight random variations — referred to as disturbances — in a manner that preserves the system's continuous dependence on the solution. The identification algorithm then determines a parameter vector p′p'p′ corresponding to the disturbed system, which differs from the original parameter vector ppp of the undisturbed system. For each parameter, we calculate the absolute values of relative deviations (RD) using the following approach: :

$$\delta i = abs((p'i - pi) / p'i); \; i=1,...,m. \quad (2)$$

For the relevant parameters, the absolute differences (pi′−pi)(p'_i - p_i)(pi′−pi) approach zero as the magnitude of the perturbations diminishes, owing to the continuous relationship between the parameters and the introduced disturbances. A similar behavior is observed for the relative deviations (RD):

$$\delta i \rightarrow 0; \; i = 1,...,n; \; \text{if disturb} \rightarrow 0. \quad (3)$$

Contrary, for the unnecessary parameters the values of the RD (2) are close to one due to (1):

$$\delta i \rightarrow 1; \; i = n+1,...,m; \; \text{if disturb} \neq 0 \quad (4)$$

Criteria (3) and (4) were derived specifically for the precision-focused model but can be generalized to apply to broader classes of mathematical models. Typically, redundant parameters are characterized by significantly larger relative deviations (RD) as per equation (2), in contrast to essential parameters. Gradual removal of these superfluous components enhances both the robustness and accuracy of the identification process. This observation is supported by multiple case studies [2], including those presented in this work. By applying the principle of model reduction, it becomes feasible to expand the model's structure systematically—assessing each new component for relevance and discarding non-essential ones accordingly.

2. Test recovery of the Lorentz Atractor

The classical equations of the Lorentz attractor (5) are convenient for testing the principle of reduction, since they allow an analytical transformation into an equivalent form (6) convenient for our identification.

$$\begin{aligned}
&dx_1/dt = 10x_2 - 10x_1; \\
&dx_2/dt = 40x_1 - x_2 - x_1x_3; \\
&dx_3/dt = -8/3 \cdot x_3 + x_1x_2; \\
&x_1(0) = 5; \; x_2(0) = 4,5; \; x_3(0) = 39,7.
\end{aligned} \quad (5)$$

$$
\begin{aligned}
& dy_1/dt = y_2; \\
& dy_2/dt = y_3; \\
& dy_3/dt = 1040 y_1 - \frac{88}{3} y_2 - \frac{41}{3} y_3 + \\
& + 11\frac{y_2^2}{y_1} + \frac{y_2 y_3}{y_1} - y_1^2 y_2 - 10 y_1^3; \\
& y_1(0) = 5; \quad y_2(0) = -5; \quad y_3(0) = 20.
\end{aligned} \tag{6}
$$

Thus, the task of recovering the exact model (6) can be formulated as follows: given a discrete signal $y_1 = x_1$, it is necessary to compute its first, second, and third derivatives. y1'= y2, y2'= y3, y3', and solve the identification problem (7).

$$
\min_{\bar{a},\bar{b}} \sum_{m=1}^{M} \left(y'_{3m} - \sum_{i,j,k=0}^{4} a_{ijk} y_{1m}^{i} y_{2m}^{j} y_{3m}^{k} - \sum_{i,j,k=0}^{4} b_{ijk} y_{1m}^{i} y_{2m}^{j} y_{3m}^{k} y_{1m}^{-1} \right)^2 \tag{7}
$$

All polynomial coefficients of the problem (7) are 50. But for the exact model (6) only 7 coefficients are required. The remaining coefficients are unnecessary.

The polynomial representation in problem (7) includes 50 coefficients. However, the accurate model described in (6) requires only 7 of them; the rest are redundant and do not contribute to the solution. The discrete signal $y_1=x_1$ was obtained by applying numerical integration to equations (5) using the Runge-Kutta method with a time step of 0.02 seconds, covering the interval from 0 to 34 seconds. Based on the resulting dataset, a fifth-degree interpolation spline was generated, and its first three derivatives were derived analytically. These computed values formed the datasets y1m, y2m, y3m, y’3m, (m=1,…,1701), which were then used for solving the identification problem (7). Subsequently, a step-by-step reduction of the coefficient arrays aijk, bijk, according to the principle of reduction, was applied. The perturbations were added to values y’3m, with a relative value of 10-5. The RD δi (2) were calculated and the element with the largest δi was deleted. The criterion for completing the reduction is the compact set of residual RD. The sign of this is the same number of the RD that are larger and smaller than the average of the remaining area. The stopping criterion for the reduction process was a compact distribution of the remaining δi\delta_iδi values. Specifically, the number of deviations above and below the average had to be roughly equal, indicating a balanced residual set.

In the reduction process, the magnitude of the RD max(δi)-min(δi), and the middle relative error model coefficients were calculated. After 43 reduction steps, there are 7 coefficients of the exact model (6) with an middle relative error 0.0016. In Fig. 1 shows a change in relative error with increasing number of reduction step.

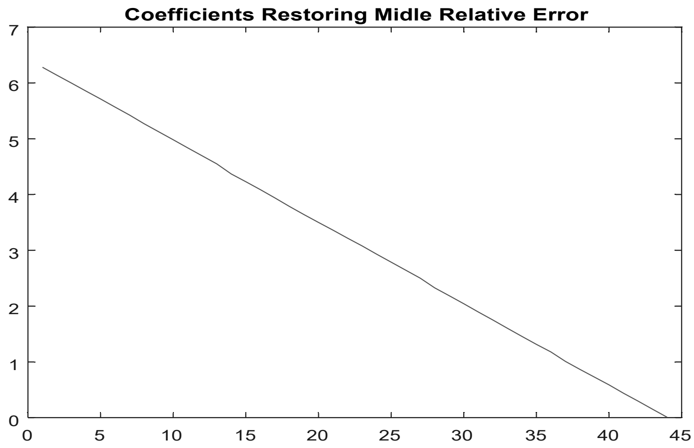

Figure 1: Changing the coefficients reproduction RD of the model (6) in the process of reduction

The dependence of the area size of relative deviations on the reduction step is shown in Fig. 2.

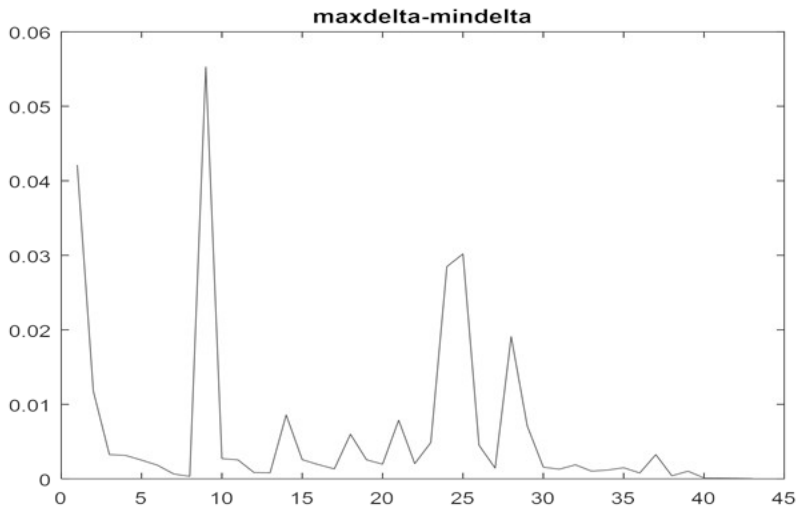

Figure 2: Change the size of the area of RD during the reduction

The same figure on an enlarged scale Fig 3. is demonstrated. The process of forming a compact area is well visible.

The model expansion approach (induction) was applied to the Lorenz system to evaluate its effectiveness. Relative deviations (as defined in equation (2)) were computed for all 50 coefficients. Beginning with the three coefficients exhibiting the lowest relative deviation values, additional coefficients were incrementally incorporated into the model—each time selecting the one with the next smallest RD from the remaining set.

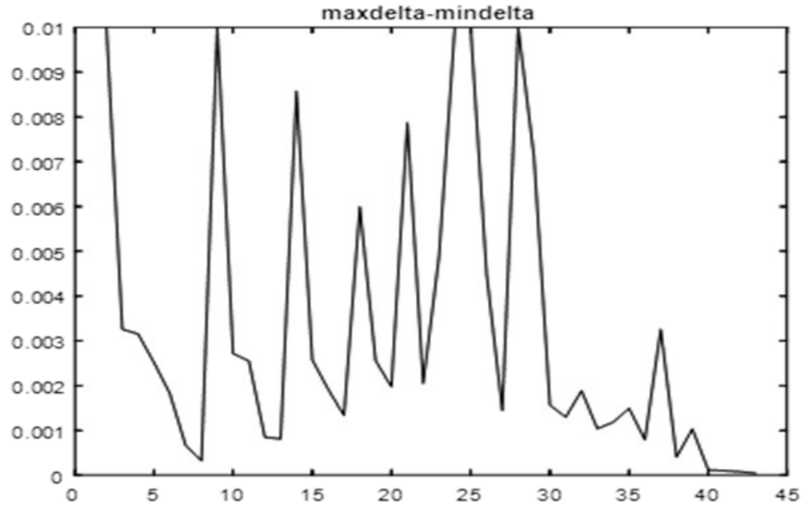

Figure 3: Change the size of the area of RD during the reduction on enlarged scale

The induction process was halted once a compact cluster of RD values was established, which occurred after four iterations.

The benefits of the induction approach over reduction are evident. Firstly, there is no need to recalculate relative deviations at every stage—initial computation suffices for the entire process. Secondly, the total number of iterations may be reduced.

Thus, using the Lorenz attractor as a test case, the fundamental principles underpinning the reduction method were effectively validated.

4. Group deletion of model elements

Removing model elements one at a time at each reduction step requires too much time to reduce complex models with a large number of elements. The reduction process can be accelerated by group removal of elements. You just need to make sure that the group you are deleting does not have the necessary model elements.

Fig. 2 shows that the size of the relative deviation region (RDR) of elements can change significantly during the reduction process. In the test case of the Lorentz attractor the necessary elements are known, and it is possible to study the change in their RD in the process of reduction. It turned out that the RD of the necessary elements is always less than the middle of the RDR at any of its sizes, The same conclusion is obtained for other test models [5].

This means that you can delete a group of elements with an RD greater than the middle of the RDR at any reduction step except for the last ones, when this criterion does not work. It is only necessary to select the minimum size of the group to be removed in order to successfully complete the reduction process.

Fig. 4 shows the change in the error of the model coefficients when deleting elements with a minimum group size of 4. The lacunaes correspond to three group deletions with group sizes of 13, 4, and 4 elements. The reduction process accelerated by 49%. If you reduce the minimum size of the deleted group to 3, the reduction process is not successful.

In the publication [6], on the basis of the principle of reduction, group removal of elements without any justification is proposed.

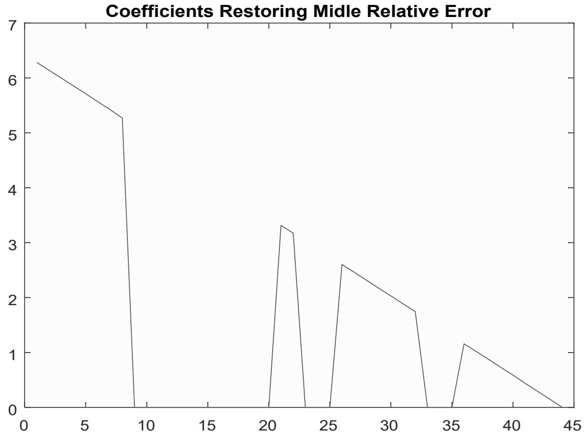

Figure 4: Changing the coefficients reproduction relative error of the model (6) in the process of groupe reduction

3.Conclusions

The improvement of mathematical models using the principle of reduction is proposed. The improvement consists in the group removal of unnecessary elements of models.

Using the example of a test model, it is shown how to form a group of deleted elements without the necessary model elements.

Declaration on Generative AI

During the preparation of this work, the authors utilised ChatGPT and LanguageTool to identify and rectify grammatical, typographical, and spelling errors. Following the use of these tools, the authors conducted a thorough review and made necessary revisions, and accept full responsibility for the final content of this publication.

References

[1] A.N.Tikhonov and V.Y.Arsenin, Solutions of Ill-Posed Problems, New York, USA: Wiley. (1977).

[2] Yann LeCun, J. S. Denker, S. Solla, R. E. Howard and L. D. Jackel, Optimal Brain Damage, Advances in Neural Information Processing Systems (NIPS 1989), 2, Morgan Kaufman, Denver, CO (1990).

[3] Yaroslav Matviychuk Dynamical Systems Mathematical Macromodelling: Theory and Practice. Lviv, Ukraine: Ivan Franko Lviv National University Publishing House. (Ukrainian). (2000) http://ena.lp.edu.ua:8080/handle/ntb/22710

[4] Matviychuk Y., Shahovska N., Kryvinska N., Poniszewska-Maranda A. New principles of finding and removing elements of mathematical model for reducing computational and time complexity // International Jornal of Grid and Utility Computing. – 2023. – Vol.14, iss.4. – P.400-410. DOI: 10.1504/IJGUC.2023.132625

[5] Yaroslav Matviychuk, Olga Karchevska, Increasing the Correctness of Mathematical Models by Novel Reduction Principle, Proceeding of the 16th International Conference on Computational Problems of Electrical Engineering, Lviv, Ukraine, Sept. 2-5. (2015) http://ieeexplore.ieee.org/document/7333351/

[6]Oleksandr Gurbych and Maksym Prymachenko. “Method for reductive pruning of neural networks and its applications”. In: Computer Systems and Information Technologies 3 (2022), pp. 40–48. doi: 10.31891/csit-2022-3-5.