Interval Mackey-Glass System in the Bipolar Coordinates*

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Abstract

Our paper is devoted to the study and design of novel chaotic systems to use in various applications. In our paper, we offer to use coordinate transformations to design a chaotic system and define its motions using algebraic-differential state space equations. We consider the known chaotic systems and apply some coordinate transformations to them. In this case, the system differential equations are used to define the known system, and observability algebraic equations depend on the used coordinate transformation. Our paper considers the transformation from cartesian coordinates into bipolar ones and vice versa. The direct transformation from cartesian coordinates in bipolar is based on using two lengths from the representative point, which define a system motion to some different base points. This transformation can be used when one interprets chaotic system state variables as coordinates in the orthogonal axes. This transformation is defined by quadratic polynomials, which usage is relatively trivial. On the contrary, the inversed transformation from bipolar to cartesian coordinates is complex enough, and its implementation can require a lot of computational resources. This drawback can be avoided by using interval methods, which allow us to define transformation equations using piecewise linear functions. In this case, one can consider the observability equations in the simplest linear-like form, which can be easily used to solve both direct and inverse transformation problems. We show the use of our approach by considering a well-known Mackey-Glass system and transforming it into bipolar coordinates by using exact and interval solutions of transformation equations. The performed research shows the similarity of the obtained results, which proves the correctness of the used approach and methods.

Keywords

Chaotic systems, coordinate transformation, interval models, direct and inverse problems

1. Introduction

The rapid growth of analog and digital communications in different areas of human activities has necessitated the development of highly secure transmission methods to protect information exchange from various unauthorized accesses and cyberattacks [1-3]. The problem becomes important due to the rise of the Internet of Things paradigm [4-6], which allows access for many users to some sensors or actuators [7]. The unauthorized person can harm such systems and cause damages in industrial and other applications [8-10].

Nowadays, different traditional cryptographic techniques are widely used, based on symmetric (AES, DES) [11-13] and asymmetric (RSA, ECC) encryption [14-17]. The main feature of these techniques is the computational complexity of mathematical problems, which are considered the basis for encryption algorithms. However, the increasing computational power of modern adversaries, which can use different application-specific integrated circuits to operate with the conventional encryption algorithm, and the potential emergence of quantum computers make conventional encryption methods vulnerable [18-19].

As a result, researchers turn their attention to alternative approaches to improve data communication security. The use of chaotic systems is one of them [20-22].

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It is well-known that chaotic systems are dynamical systems that motions are sensitive to initial conditions. Also, one can find the aperiodicity and high complexity of such systems. These facts make chaotic systems well-suited to use in secure communication applications to generate unpredictable signals that can be used for data encryption, key generation, and establishing secure data transmission channels with high resistance to brute-force attacks. [23-25]

Moreover, since chaotic systems can be implemented using digital and analog devices, they can be used in various transmission environments and establish secured optical, radio-frequency, and acoustic data communication channels. [26-27]

Many papers are devoted to studying known and designing novel chaotic systems. The main drawback of known papers is the very subjective design of chaotic systems. Authors start considering a system without explaining why and how this system is designed. This fact makes system improvement a pretty challenging process. [28-30]

We offer to avoid this drawback by applying some coordinate transformation to known chaotic systems and designing a novel system with known features on their basis.

Our paper is organized as follows. At first, we consider the generalized first-order delayed nonlinear system and apply to it coordinate transformation, which defines its motion in the bipolar coordinates. Then, we consider an inverse transformation and show its complexity. Third, we show the use of interval methods to simplify the chaotic system transformation. We consider using our approach by transferring a well-known Mackey-Glass system with exact and interval transformations.

2. Method

2.1. The Exact Direct and Inverse Transformations Between the Cartesian and Bipolar Coordinates

It is a well-known fact that motion of some dynamical system can be studied by using its phase portrait (Figure 1).

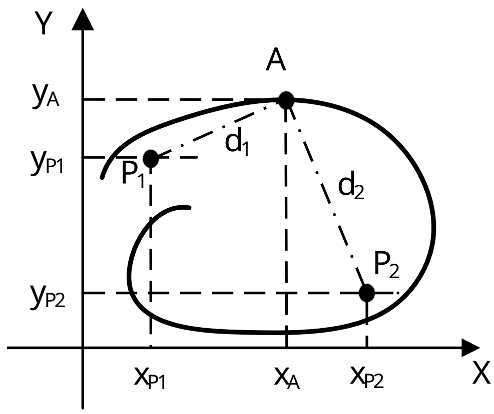


Figure 1: System phase portrait

In general case this phase portrait represents 2D projection of system motions which are produced as solution of nonlinear differential equations. In our paper we consider a first-order delayed nonlinear dynamical system which motion is given as follows

$$\dot{x} = f(x, x_{tau}) \tag{1}$$

where x is a system generalized state variable, f(.) is some nonlinear function and τ is a system time delay.

We assume that all system motions are localized in first quadrant of system phase plane as it is shown in Figure 1 and we consider the horizontal axis as axis where current values of system state variable are placed and vertical axis is used as position of delayed values of the state variable. It is clear that both of these coordinates are defined in relation to axes origin and it is necessary to have the possibility to define a distance between the origin and corresponding projection of system representative point A to find the system coordinate.

At the same time the analysis of system phase portrait and applying various coordinate transformations to it gives us the possibility to design novel dynamical systems. We claim that such an approach is based on the possibility of different interpretation of system state variables. Due to the using of coordinate transformations the resulted system can produce the desired motions in the given phase plane's domain.

In our paper we offer to consider the system motion in bipolar coordinates. In this case the position of system representative point A can be defined by using distances d_1 and d_2 from some base points P_1 and P_2 to the considered representative point. Here we assume that coordinates x_{P1}, x_{P2}, yP1, y_{P2} of P_1 and P_2 points are known.

If one take into account Figure 1 he can write down expressions which define the distances between points in such a way

$$d_1^2=(x_A-x_{P1})^2+(y_A-y_{P1})^2; d_2^2=(x_A-x_{P2})^2+(y_A-y_{P2})^2, x_A=x; y_A=x_\tau \tag{2}$$

From the control theory viewpoint one can consider (2) as polynomial observability equations for system (1). These equations allow us to solve both direct and inverse problems of system coordinate transformation. The direct problem gives us the possibility to define system position in bipolar coordinates with pair of distances d_1 and d_2 by using known system state variable and base points. It is clear that due to the quadratic functions in (2) the solution of the direct problem allows to produce direct d_{1p}, d_{2p}, and inverse d_{1m}, d_{2m} system outputs

$$\begin{gathered} d_{1p}=\sqrt{(x_A-x_{P1})^2+(y_A-y_{P1})^2}; d_{1m}=-\sqrt{(x_A-x_{P1})^2+(y_A-y_{P1})^2}; \\ d_{2p}=\sqrt{(x_A-x_{P2})^2+(y_A-y_{P2})^2}; d_{1m}=-\sqrt{(x_A-x_{P2})^2+(y_A-y_{P2})^2} \end{gathered} \tag{3}$$

which can be used in various applications.

If one solves (2) for system coordinates in the cartesian plane he obtains the solution of inverse problem and define the position of system representative point as function of base points coordinates and distances between these points and representative point

$$\begin{gathered} x_A=x_{P1}\pm\frac{2(y_{P1}-y_{P2})}{(2(x_{P1}-x_{P2})^2+2(y_{P1}-y_{P2})^2)}\begin{pmatrix} d_1^2-d_2^2+(x_{P1}-x_{P2})^2+(y_{P1}-y_{P2})^2+ \\ +(y_{P1}-y_{P2})^2 \end{pmatrix}\times \\ \sqrt{\begin{matrix}(x_{P1}-x_{P2})^2-(d_1-d_2-y_{P1}+y_{P2})(d_1-d_2+y_{P1}-y_{P2})(x_{P1}-x_{P2})^2\times \\ \times(-(x_{P1}-x_{P2})^2+(d_1+d_2)^2-(y_{P1}-y_{P2})^2)\end{matrix}}-y_{P1}^6+ \\ +(2d_1^2+2d_2^2-(x_{P1}-x_{P2})^2-15y_{P2}^2)y_{P1}^4-8\left(d_1^2+d_2^2-\frac{(x_{P1}-x_{P2})^2}{2}-\frac{5}{2}y_{P2}^2\right)y_{P2}y_{P1}^3\cdot \\ -15y_{P2}^4+(12d_1^2+12d_2^2-6(x_{P1}-x_{P2})^2)y_{P2}^2+x_{P1}^4-4x_{P1}^3x_{P22}+6y_{P1}^5y_{P2}+ \\ +(4d_1^2+6x_{P2}^2)x_{P1}^2+(-8d_1^2x_{P2}-4x_{P2}^3)x_{P1}+x_{P2}^4+4d_1^2x_{P2}^2-(d_1-d_2)^2(d_1+d_2)^2y_{P1}^{¿} \end{gathered} \tag{4}$$

$$y_A = \frac{\begin{array}{c}(d_1^2 - y_{P1}^2)(y_{P2} - y_{P1}) + (d_2^2 - y_{P2}^2)(y_{P1} - y_{P2}) + \\ + y_{P1}(x_{P1} - x_{P2})^2 + y_{P2}(x_{P1} - x_{P2})^2\end{array}}{(2(x_{P1} - x_{P2})^2 + 2(y_{P1} - y_{P2})^2)} \pm$$

$$\pm \frac{\sqrt{\begin{array}{c}-(d_1^2 - d_2^2)^2(x_{P1} - x_{P2})^2 + 2d_1^2(x_{P1} - x_{P2})^4 + 4d_1^2(x_{P1} - x_{P2})^2(y_{P1} - y_{P2})^2 + \\ + 2d_2^2(x_{P1} - x_{P2})^4 + 2d_2^2 x_{P1}^2 y_{P1}^2 - 4d_2^2 x_{P1}^2 y_{P1} y_{P2} - 4d_2^2 x_{P1} x_{P2} y_{P1}^2 + \\ + 8d_2^2 x_{P1} x_{P2} y_{P1} y_{P2} + 2d_2^2 y_{P2}^2 (x_{P1} - x_{P2})^2 + 2d_2^2 x_{p2}^2 y_{p1}^2 - \\ - 4d_2^2 x_{P2}^2 y_{P1} y_{P2} - x_{P1}^6 + 6x_{P1}^5 x_{P2} - 15x_{P1}^4 x_{P2}^2 - 2x_{P1}^4 y_{P1}^2 + 4x_{P1}^4 y_{P1} y_{P2} - \\ - 2x_{P1}^4 y_{P2}^2 + 20x_{P1}^3 x_{P2}^3 + 8x_{P1}^3 x_{P2} y_{P1}^2 - 16x_{P1}^3 x_{P2} y_{P1} y_{P2} + 8x_{P1}^3 x_{P2} y_{P2}^2 - \\ - 15x_{P1}^2 x_{P2}^4 - 12x_{P1}^2 x_{P2}^2 y_{P1}^2 + 24x_{P1}^2 x_{P2}^2 y_{P1} y_{P2} - x_{P2}^2 y_{P2}^4 - \\ - 12x_{P1}^2 x_{P2}^2 y_{P2}^2 - x_{P1}^2 y_{P1}^4 + 4x_{P1}^2 y_{P1}^3 y_{P2} - 6x_{P1}^2 y_{P1}^2 y_{P2}^2 + \\ + 4x_{P1}^2 y_{P1} y_{P2}^3 - x_{P1}^2 y_{P2}^4 + 6x_{P1} x_{P2}^5 + 8x_{P1} x_{P2}^3 y_{P1}^2 - 16x_{P1} x_{P2}^3 y_{P1} y_{P2} + \\ + 8x_{P1} x_{P2}^3 y_{P2}^2 + 2x_{P1} x_{P2} y_{P1}^4 - 8x_{P1} x_{P2} y_{P1}^3 y_{P2} + 12x_{P1} x_{P2} y_{P1}^2 y_{P2}^2 - \\ - 8x_{P1} x_{P2} y_{P1} y_{P2}^3 + 2x_{P1} x_{P2} y_{P2}^4 - x_{P2}^6 - 2x_{P2}^4 y_{P1}^2 + 4x_{P2}^4 y_{P1} y_{P2} - \\ - 2x_{P2}^4 y_{P2}^2 - x_{P2}^2 y_{P1}^4 + 4x_{P2}^2 y_{P1}^3 y_{P2} - 6x_{P2}^2 y_{P1}^2 y_{P2}^2 + 4x_{P2}^2 y_{P1} y_{P2}^3\end{array}}}{2(x_{P1} - x_{P2})^2 + 2(y_{P1} - y_{P2})^2}$$

One can consider (4) as observability equation for the nonlinear system (1) in the case when system state variables are considered as distances d_1 and d_2 between points in the phase plane. Due to the solution of system quadratic equations one gets several signals in this case as well. It is necessary to say that the use of quadratic dependencies like (2) to define representative point's position cause a quite complex solution of inverse problem which requires some calculation resources to define a system position during its study and implementation.

It is necessary to say that the solution of inverse problem can be used to redefine (1) in terms d_1 and d_2 only by substituting (4) into (1). Such an approach allows us to exclude from a consideration the observability equations (3). At the same time the resulting motion equation can become a quite complex and hardly have a practical usage.

2.2. The Interval Direct and Inverse Transformations Between the Cartesian and Bipolar Coordinates

We offer to avoid the above-mentioned drawback by replacing the exact nonlinear functions in the right-hand expressions of (2) with intervals of their possible values. Due to the similarity of summands in (2) we use the simplified transformation of nonlinear function into the interval form by replacing each nonlinear summand with intervals of their possible values. In this case we consider the nonlinear function as one variable functions. Although, one can use the same approach in the most general case of multivariable functions by studying their boundaries in the state space.

Let us consider the proposed approach by using the following quadratic nonlinear function

$$f_1(x) = (x - x_P)^2 \tag{5}$$

This function can be bounded by some piecewise linear functions $f_{1max}(x)$ and $f_{1min}(x)$

$$f_{1min}(x) = \begin{cases} a_{11min}x + b_{11min} & if \quad x_{1min} \le x < x_{11}; \\ a_{1nmin}x + b_{1nmin} & if \quad x_{1n} < x \le x_{1max}, \end{cases} \tag{6}$$

$$f_{1max}(x) = \begin{cases} a_{11max}x + b_{11max} & if \quad x_{1min} \le x < x_{11}; \\ a_{1nmax}x + b_{1nmax} & if \quad x_{1n} < x \le x_{1max}, \end{cases}$$

here a_{ijmin}, a_{ijmax}, b_{ijmin}, b_{ijmax} are piecewise linear factors, x_{ij} are fracture points where piecewise linear function change its parameters, n and m are numbers of fracture points in upper and lower boundaries.

It should be mentioned that in the most general case the fracture points for lower and upper boundaries can be different. One can define these points as the solution of optimization problem

$$I=\int_{x_{min}}^{x_{max}} (f_{1min}(x)-f_1(x))^2+(f_{1max}(x)-f_1(x))^2 dx \rightarrow min \tag{7}$$

for the given numbers n and m of the fracture points.

Since the considered boundary functions are piecewise linear one, (7) can be rewritten as follows

$$I=\sum_{i=0}^{n}\int_{x_i}^{x_{i+1}} (f_{1min}(x)-f_1(x))^2 dx+\sum_{i=0}^{n}\int_{x_i}^{x_{i+1}} (f_{1max}(x)-f_1(x))^2 dx \rightarrow min \tag{8}$$

here $x_0=x_{min}$ and $x_{m+1}=x_{max}$.

In other word the problem of definition of fracture points can be considered as the problem of minimization the square of domains between the considered function and its boundaries.

We use (6) as boundaries to define the interval of possible values of function f(x)

$$f_1(x)\in \boldsymbol{f_1(x)};\boldsymbol{f_1(x)}=[f_{1min}(x),f_{1max}(x)] \tag{9}$$

The graphical representation of our approach is shown in Figure. 2.

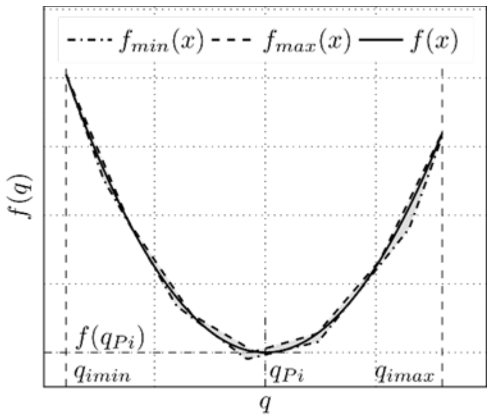


Figure 2: Piecewise boundaries for the quadratic nonlinearity

As one can see the considered nonlinear function belongs to the defined interval and it does not exceed interval (7) for any values from the interval of possible values of system state variable.

This fact proves the correctness of the used approach to replace the nonlinear function with the linear or piecewise linear intervals which define the domain where the initial nonlinear function is defined.

Substitution (6) into (9) gives us the possibility to define the domain ***f₁(x)*** in terms of function argument x and approximation parameters a_i and b_i

$$\begin{gathered}\boldsymbol{f_1=a_1x_1+b_1=[a_{1min},a_{1max}]x_1+[b_{1min},b_{1max}]},\\ \boldsymbol{a_{1min}=\cup_{i=1}^{n1}a_{1imin};a_{1max}=\cup_{i=1}^{m1}a_{1imax};b_{1min}=\cup_{i=1}^{n1}b_{1imin};b_{1max}=\cup_{i=1}^{m1}b_{1imax}},,\\ \boldsymbol{a_{1imin}=[a_{1imin},a_{1(i+1)min}],b_{1imin}=[b_{1imin},b_{1(i+1)min}]},\\ \boldsymbol{a_{1imax}=[a_{1imax},a_{1(i+1)max}],b_{1imax}=[b_{1imax},b_{1(i+1)max}]}.\end{gathered} \tag{10}$$

Thus, the use of interval methods gives us the possibility to redefine the initial nonlinear function in the linear-like interval form which can be easy used to solve various mathematical problems.

It is clear that the above-given transformations can be performed for the second summand in (2) as well

$$f_2(x) \in \boldsymbol{f_2(x)}, \boldsymbol{f_2} = \boldsymbol{a_2 x_2} + \boldsymbol{b_2} = [\boldsymbol{a_{2min}}, \boldsymbol{a_{2max}}]\boldsymbol{x_2} + [\boldsymbol{b_{2min}}, \boldsymbol{b_{2max}}], \tag{11}$$
$$\boldsymbol{a_{2min}} = \cup_{i=1}^{n2} \boldsymbol{a_{2imin}}; \boldsymbol{a_{2max}} = \cup_{i=1}^{m2} \boldsymbol{a_{2imax}}; \boldsymbol{b_{2min}} = \cup_{i=1}^{n2} \boldsymbol{b_{2imin}}; \boldsymbol{b_{2max}} = \cup_{i=1}^{m2} \boldsymbol{b_{2imax}},,$$
$$\boldsymbol{a_{2imin}} = [\boldsymbol{a_{2imin}}, \boldsymbol{a_{2(i+1)min}}], \boldsymbol{b_{2imin}} = [\boldsymbol{b_{2imin}}, \boldsymbol{b_{2(i+1)min}}],$$
$$\boldsymbol{a_{2imax}} = [\boldsymbol{a_{2imax}}, \boldsymbol{a_{2(i+1)max}}], \boldsymbol{b_{2imax}} = [\boldsymbol{b_{2imax}}, \boldsymbol{b_{2(i+1)max}}].$$

We define boundaries $f_{2min}(x)$ and $f_{2max}(x)$ by using expression which is similar to (6) but defined by its own piecewise linear approximation factors. Also, we assume that number of fracture points n_i and m_i are different for different boundaries.

If one substitutes (11) and (10) into (2) he can rewrite the last equations as follows

$$\boldsymbol{d_1^2} = [d_{1min}^2, d_{1max}^2] = \boldsymbol{a_{11} x_A} + \boldsymbol{a_{12} y_A} + \boldsymbol{b_{11}} + \boldsymbol{b_{12}};$$
$$\boldsymbol{d_2^2} = [d_{2min}^2, d_{2max}^2] = \boldsymbol{a_{21} x_A} + \boldsymbol{a_{22} y_A} + \boldsymbol{b_{21}} + \boldsymbol{b_{22}}. \tag{12}$$

here interval factors $\boldsymbol{a_{ij}}$ and $\boldsymbol{b_{ij}}$ are caused by using coordinates of different base points.

We call (12) as interval direct observability equations which allows to define the distance between the system representative point and some base point. Analysis of right-hand expressions in (12) shows these expressions are linear for system state variables.

Similar to (3) one can consider the solution (12) for d_i as the source of two inverted signals. Contrary to (3) one should take into consideration that these signals are defined in the interval form and rather define domain where signals are localized than the signals. Although the length of this domain can be considered neglectable small in case of small intervals (11) and (12). In this case (12) can be considered as almost exact solution of the direct problem.

In the most general case, one can use different ways to localize the considered signals. In our paper we offer to use sliding mode approach to perform such an operation

$$d_i = \pm\left(\frac{(d_{imin} + d_{imax})}{2} + \frac{(d_{imax} - d_{imin})}{2} g([S]) sign(S)\right). \tag{13}$$

here S is equation of some sliding plane which define the switching between upper and lower boundaries of the defined distance and g(.) is some odd function which allows to control the amplitude of sliding mode oscillations.

Thus, the use of interval methods allows us to simplify the solution of the direct problem by using linear expressions in system observability equations. This approach gives the possibility to reduce the calculation resources, which are necessary to spend, to define the considered system outputs in case of using low and middle-range MCU without hardware multiplication support.

We think that the main benefits of using (12) instead of (2) is the possibility to solve the inverse problem in a quite simple way

$$\boldsymbol{x_A} = \frac{\boldsymbol{a_{22} d_1^2} - \boldsymbol{a_{12} d_2^2} + (\boldsymbol{b_{21}} + \boldsymbol{b_{22}})\boldsymbol{a_{12}} - (\boldsymbol{b_{11}} + \boldsymbol{b_{12}})\boldsymbol{a_{22}}}{\boldsymbol{a_{11} a_{22}} - \boldsymbol{a_{12} a_{21}}};$$
$$\boldsymbol{y_A} = \frac{\boldsymbol{a_{11} d_2^2} - \boldsymbol{a_{21} d_1^2} - (\boldsymbol{b_{21}} + \boldsymbol{b_{22}})\boldsymbol{a_{11}} + (\boldsymbol{b_{11}} + \boldsymbol{b_{12}})\boldsymbol{a_{21}}}{\boldsymbol{a_{11} a_{22}} - \boldsymbol{a_{12} a_{21}}}. \tag{14}$$

Comparison of (14) and (4) proves our claiming about simplification of the inverse problem's solution. Although, it is clear that the determination of system state variables according to (14) requires to know the intervals $\boldsymbol{a_{ij}}$ and $\boldsymbol{b_{ij}}$. That is why one should define the intervals of system state variable $[x_i, x_{i+1}]$ and $[y_i, y_{i+1}]$ in which equations (12) have different signs. Approximation factors for these intervals can be used in (14) to define system state variables by known distances from the base points.

3. Results and Discussion

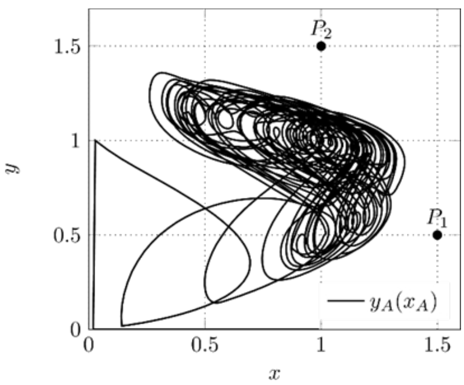
3.1. Exact Mackey-Glass System in the Bipolar Coordinates

We show the use of our approach to define system dynamic in bipolar coordinates by considering a well-known Mackey-Glass equation

$$\dot{x}_A = -\gamma x_A + \beta \frac{y_A}{1} + y_A^n, y_A = x_{A\tau}. \tag{15}$$

here γ and β are system factors, n means some power, x_A is a state variable that shows position of representative point A in the horizontal axis of system phase plane (Figure. 3), we use the shifted in τ sec value of the state variable to define the system vertical position in phase plane.

Under some parameters and initial conditions (15) define chaotic system motions (Figure.4) and can be considered as the mathematical basis to design a chaotic generator.



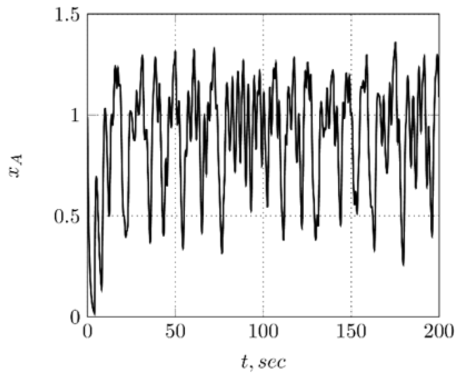


Figure 3: Mackey-Glass system phase plane **Figure 4:** Mackey-Glass motion trajectory

Here we study system with following parameters γ=1, β=2, τ=4, n=10, x(0)=1.

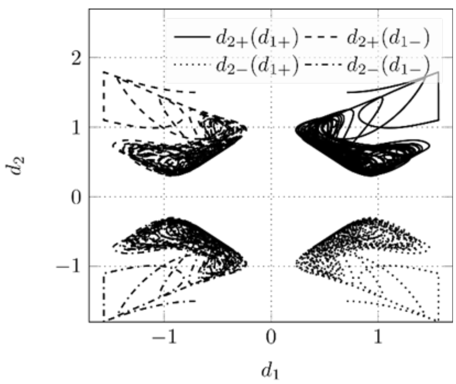
Since the Mackey-Glass system is a well-studied dynamical system its usage as true random generator in various applications which are connected with secured data transmission can be compromised and cause data leakage.

That is why we offer to use some novel coordinate basis where motion of this system is not known yet. We consider bipoloar coordinate system and assume that in the system phase plane two base points P_1 and P_2 are defined (Figure.3).

In this case usage (3) allows us to perform direct transformation from cartesian coordinates into bipolar one. It is possible to claim that such a transformation defines motions of the novel chaotic systems by using (15) and (3). Systems motions as wells as its phase portrait are shown in Figure.5 and Figure.6.

Analysis of the given in Figure.5 -Figure.6 simulation results for Mackey-Glass system in bipolar coordinates proves the possibility to design novel chaotic system by applying some nonlinear coordinate transformations to known chaotic system. Such a coordinate transformations lead to to considering nonlinear observability equations which define only system output but do not change its inner dynamic. Thus, if the initial system moves through the chaotic trajectories, the transformed one also have a chaotic nature.

At the same time, the usage of proposed approach, which is based on the determination of distance between system representative point and some base points, allows us to define new system outputs which number equals to number of base points which are used to define system coordinates. Thus, one can use this fact to increase the number of system outputs which produce chaotic oscillations. Comparison the oscillations in Figure.4 and Figure.6 allows us to claim that the use of nonlinear transformations allows us to change the form and frequency of oscillations as well as system attractors.



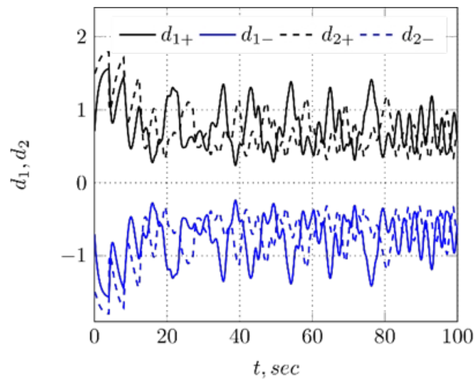
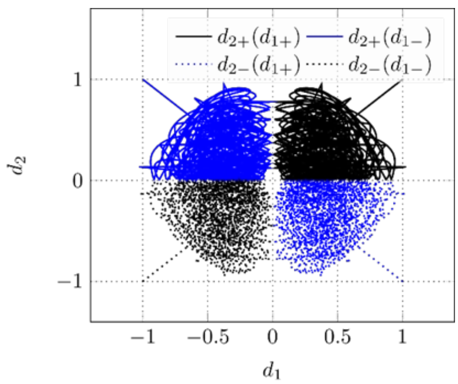


Figure 5: Mackey-Glass attractor in bipolar **Figure 6:**
Mackey-Glass trajectories in coordinates bipolar coordinates

The one more feature of the considered approach is the use of quadratic polynomials which solution allows to define both positive and negative values for the system coordinates. Since these coordinates are considered as distances between points, we call the case of positive distances as the main one and cases with one or two negative coordinates are considered as secondary. This fact defines four possible system attractors which are shown in Figure.6 and one can use different switching techniques to switch from one attractor to another and design variable structure chaotic systems which design is out of our paper's scope.

In Figure.5 and Figure.6 we show the simulation results for the chaotic system with fixed coordinates of the base points. We see one more way to improve system features by considering its motions relatively to moved base points. It is clear that the base points' motions can be defined by using different laws and it also require a detailed study which is a topic of our future research.

Here we show the principle of changing system dynamic by considering coordinates of base points as the values of system state variable x_A which are taken in different time moments. In Figure.7 and Figure.8 simulation results are shown for the case when $x_{P1}=x_{A1}$, $y_{P1}=x_{A2}$, $x_{P2}=x_{A3}$, $y_{P2}=x_{A4}$, here number near x_A variable means number of seconds to shift the signal.



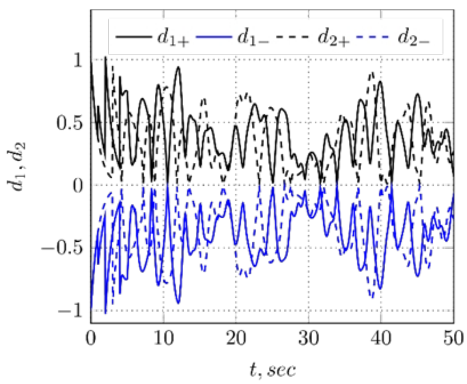


Figure 7: Mackey-Glass attractor in bipolar coordinates with moved base points

Figure 8: Mackey-Glass trajectories in bipolar coordinates with moved base points

Analysis of the given in Figure.7 and Figure.8 curves shows that the positions of base points make great effect in system dynamic and allows to design one more novel chaotic system which attractor is localized in the given quadrant and has form different from the above-considered attractors.

All above-studied models are designed for the case when system state variables are considered as coordinates in some cartesian system. At the same time, one can consider state variables of any dynamical system without any references to coordinate systems. For example, one can interpret system state variables as given in the bipolar coordinate system. In this case transformation of Mackey-Glass system dynamic is performed by using the inversed observability equations (4).

Simulation results for the Mackey-Glass system with observability equations (4) are shown in Figure.9 and Figure.10

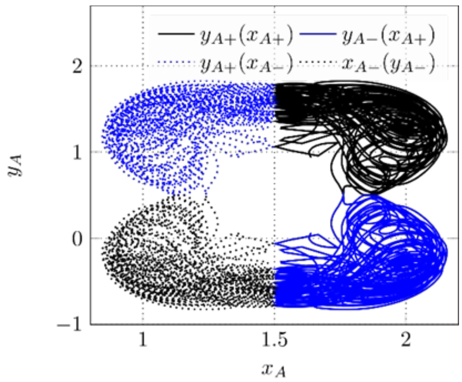


Figure 9: Transformed Mackey-Glass system attractor

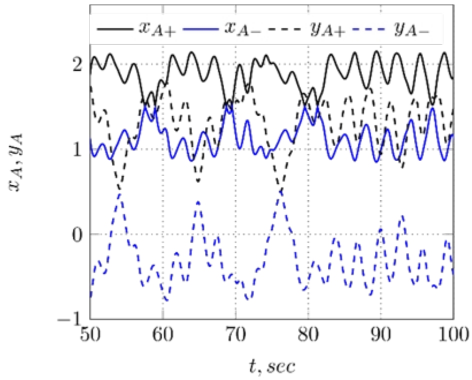


Figure 10: Transformed Mackey-Glass system trajectories

Analysis of the given in Figure.9 and Figure.10 simulation results proves the possibility to change a system output trajectory as well as its phase portrait by using nonlinear observability equations. Similar to previous considered systems, dynamical system from Figure.9 and Figure.10 defines two pair of output signals. Each pair are defined by direct and inversed signals which can be used in various applications. At the same time, one should take into account that combinations of distances d_1 and d_2 should correspond the base point coordinates. In other words, not all bipolar coordinates d_1 and d_2 can be transformed into the cartesian one. The possibility to perform transformation can be found from (4) which should take real values. The complex values of system output show that such a combination of bipolar coordinates cannot be transformed into cartesian ones.

All above-studied systems have symmetric attractors and can be used as the basis to design more complex chaotic systems. However, if one assumes that base points are moved in the phase plane and their motion is defined according above-shown interrelations, one can get more complex chaotic motion (Figure.11 and Figure.12)

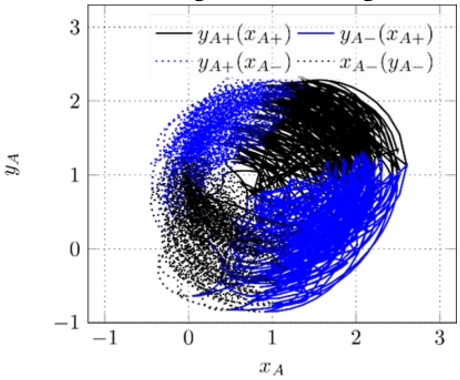


Figure 11: Transformed Mackey-Glass system attractor with chaotically moved base points

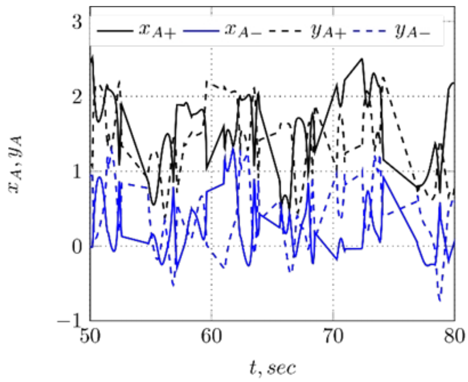


Figure 12: Transformed Mackey-Glass system trajectories with chaotically moved base points

The given in Figure.11 and Figure.12 simulation results shows that in the most common case the produced system outputs in each pair can be asymmetrical ones. As the result attractors overlapping can be found. One can use this fact to design highly nonlinear multichannel chaotic system which outputs are non-inverse and produced according non-harmonic dependencies.

3.2. Interval Mackey-Glass System in the Bipolar Coordinates

Due to the complexity of the above-used transformations, now we consider the Mackey-Glass system and its coordinate transformations in the interval form.

It is clear that the main feature of the considered system, which cause the chaotic dynamic, is nonlinear function of delayed state variable.

Let us rewrite this function in the interval form (9) and define boundary functions as follows

$$f_{1min}(y_A)=\begin{cases} y_A & if \quad 0<y_A<0.68 \\ 0.43\,y_A+0.3 & if \quad 0.68<y_A<0.8; \\ -0.43\,y_A+1.08 & if \quad 0.8<y_A<0.89; \\ -1.86\,y_A+2.35 & if \quad 0.89<y_A<1.18; \\ -0.54\,y_A+0.79 & if \quad 1.18<y_A<1.44; \\ -0.01\,y_A+0.031 & if \quad 1.44<y_A<1.9, \end{cases} \tag{15}$$

$$f_{1max}(y_A)=\begin{cases} y_A & if \quad 0<y_A<0.65 \\ 0.58\,y_A+0.26 & if \quad 0.65<y_A<0.78;; \\ -0.41\,y_A+1.04 & if \quad 0.78<y_A<0.88; \\ -1.66\,y_A+2.15 & if \quad 0.88<y_A<1.19; \\ -0.61\,y_A+0.91 & if \quad 1.19<y_A<1.4; \\ -0.08\,y_A+0.161 & if \quad 1.4<y_A<1.9, \end{cases}$$

Graphically these functions are given in Figure.13

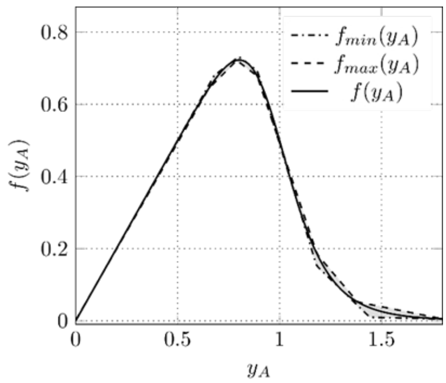


Figure 13: Piecewise-linear interval approximation of the Mackey-Glass nonlinearity

Simulation results for such a system are shown in Figure.14 and Figure 15.

As one can see the use of interval methods allows us to define interval signal which is bounded by lower and upper boundary system trajectories. At the same time, the system attractor as well as its oscillations are similar to the initial nonlinear system. This fact allows us to claim correctness of the performed transformation from exact to interval system. Such a transformation extended the class of chaotic systems and allows to consider the systems which produce an infinite set of signals from the filled domain (Figure.15). One can use various methods to select one or several exact signals from this set.

If take into account the coordinates of the base points he can perform the piecewise linear interval approximation (2) and transform the nonlinear observability equations into linear-like interval form (10).

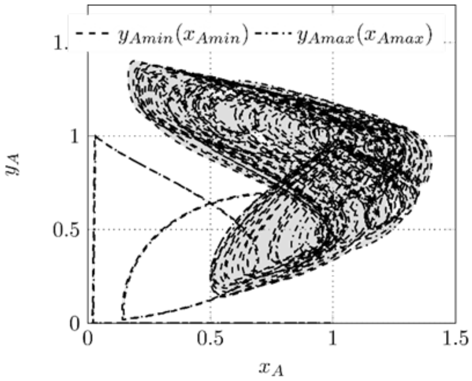


Figure 14: Piecewise linear Mackey-Glass system interval attractor

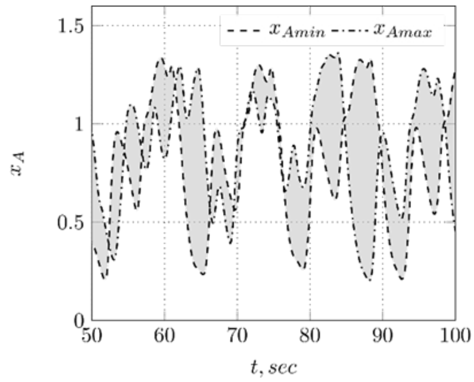


Figure 15: Piecewise linear Mackey-Glass system interval motion trajectories

Simulation results for interval Mackey-Glass system (15) with boundaries (16) and observability equations (10) are shown in Figure.16 and Figure.17

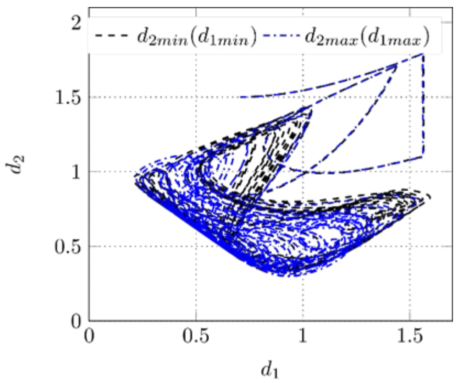


Figure 16: Piecewise linear Mackey-Glass system interval attractor in bipolar coordinates

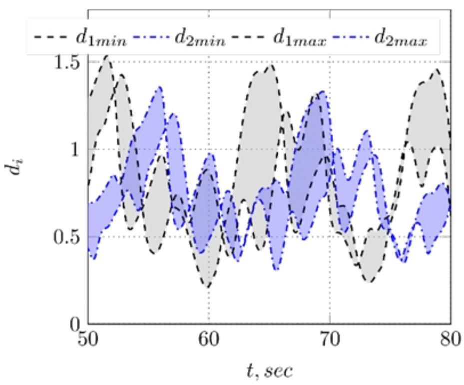


Figure 17: Piecewise linear Mackey-Glass system interval trajectories in bipolar coordinates

Comparison of attractors in Figure 16 and Figure.5 show their similarity that prove the possibility to use the interval methods to design and study systems with chaotic dynamic. Contrary to the classical approach which is based on the use of nonlinear transformation equations the use of piecewise linear interval equations allows to simplify the mathematical model of the considered system and reduce the computational resources which are used to simulate system in bipolar coordinates with fixed base points. It is clear that it becomes possible because of the one-time solving optimization problem (8) which can be done before simulation starting. At the same time, in case of the use moved base points piecewise linear approximation should be performed for all base points coordinates.

Nevertheless, the all-above-given features of system with interval piecewise linear observability equation, the main benefit of such systems is the possibility to perform inverse transformation from bipolar coordinates into cartesian ones without performing complex calculations. In Figure.18 and Figure 19 we show the simulation results for the case when the system state variables are considered as bipolar coordinates and then applying (14) allows us to solve the inverse problem.

Comparison attractors in Figure.9 and Figure. 18 shows the similar patterns which proves the correctness of performed transformation. Moreover, the considering interval model allows us to study the attractor's deformation which are caused by motions in various system boundaries. This study allows us to claim that real system attractor is in the domain where both minimal and maximal attractors are defined. Also, one can see that approaching the upper boundary compresses the system attractor.

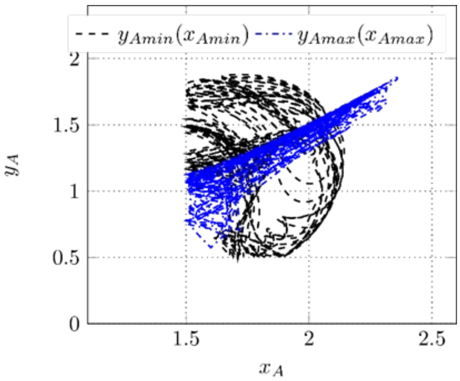


Figure 18: Piecewise linear Mackey-Glass system interval attractor in cartesian coordinates coordinates

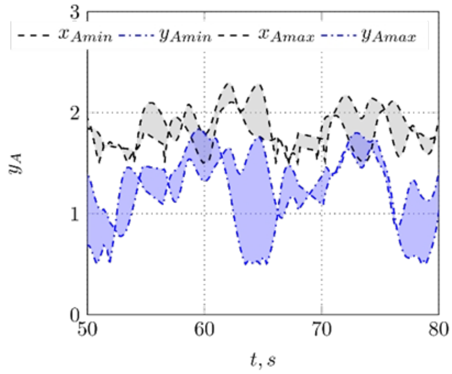


Figure 19: Piecewise linear Mackey-Glass system interval motion in cartesian

4. Conclusion

Our studies claim that using nonlinear coordinate transformation equations as observability equations allows us to design novel chaotic systems as systems which dynamic are defined by state-space equations. The features of this system depend on both the initial chaotic system and the coordinate transformation used. Since these transformations can be quite complex, it can be hard to use them to solve direct and inverse transformation problems. One can avoid this drawback by using interval methods, which allows us to rewrite system dynamics as piecewise linear differential equations with the piecewise linear observability equations. Such an approach makes it possible to get results similar to the solution of exactly defined equations.

We see the future development of our research in designing the control system to synchronize the considered chaotic systems with some signals.

Declaration on Generative AI

The author(s) have not employed any Generative AI tools..

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