The Problem of Possibility-Probability Optimization with Constraints on the Possibility/Necessity-Probability and Probability-Possibility/Necessity

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Abstract
The paper continues the study of the problem of possibilistic-probabilistic linear programming with constraints on the possibility/necessity - probability and probability - possibility/necessity. For the case of normally distributed random and fuzzy factors in the shift-scale representation of fuzzy random variables, an equivalent stochastic analog of the model is constructed. The obtained results are applied to the construction of a minimal risk portfolio in the conditions of hybrid uncertainty.

Keywords
possibilistic- probabilistic optimization, constraints on possibility/necessity and probability, constraints on probability and possibility/necessity, equivalent deterministic analog, equivalent stochastic analog, fuzzy random variable, strongest t-norm, minimum risk portfolio

1. Introduction

The proposed article continues the research started in [1] and is devoted to the problem of optimization in conditions of hybrid uncertainty of the possibility-probability type. Its results can be used in the construction and research of generalized models of portfolio analysis, in economic and mathematical planning and other areas.

The developed indirect methods for solving problems of this class, considered in the context of the strongest t-norm describing the interaction of fuzzy parameters, are based on the construction of their equivalent deterministic analogues. For this purpose, two-level procedures are used to remove uncertainty of probabilistic and fuzzy types, based on the principles of expected possibility [2-9,24] and the implementation of restrictions on possibility/necessity and probability.

In contrast to [1], where the model is studied under the constraints of possibility/necessity-probability, in this paper we study the problem of possibilistic-probabilistic linear programming under the constraints of probability-possibility/necessity. An equivalent stochastic analog of the model is constructed in the class of normal probability distributions that characterize the fuzzy parameters of the model, and for normally distributed random parameters in the shift-scale representation of fuzzy random variables. The properties of the constructed equivalent model are investigated. The obtained results are applied to the construction of a minimal risk portfolio model.

2. Necessary concepts and notations

In the context of works [10-15], we introduce a number of definitions and concepts from the theory of possibilities. Let further $(\Gamma, \mathrm{P}(\Gamma), \tau)$ and $(\Omega, \mathrm{B}, \mathbf{P})$ be possibility and probability spaces, respectively, in which Ω is the space of elementary events $\omega \in \Omega$, Γ is the model space with elements $\gamma \in \Gamma$, B is the

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σ-algebra of events, $P(\Gamma)$ is the set of all subsets of Γ, $\tau \in \{\pi, \nu\}$, π and ν are measures of possibility and necessity, respectively, and **P** is the probability measure; $\mathbb{E}^1$ is the number line.

Definition 1. Fuzzy random variable $Y(\omega,\gamma)$ is a real function $Y: \Omega \times \Gamma \to \mathbb{E}^1$ σ-measurable for each fixed γ, and

$$\mu_Y(\omega,t) = \pi\{\gamma \in \Gamma: Y(\omega,\gamma) = t\}, \forall t \in \mathbb{E}^1$$

is called its distribution function.

From definition 1 it follows that the distribution function of a fuzzy random variable depends on a random parameter, i.e. it is a random function.

Definition 2. Let $Y(\omega, \gamma)$ – be a fuzzy random variable. Its expected value $\mathbf{E}[Y]$ is a fuzzy value that has a possibilities distribution function

$$\mu_{\mathbf{E}[Y]}(t) = \pi\{\gamma \in \Gamma: \mathbf{E}[Y(\omega,\gamma)] = t\}, \forall t \in E^1,$$

where **E** — is a mathematical expectation operator

$$\mathbf{E}[Y(\omega,\gamma)] = \int_\Omega Y(\omega,\gamma)\mathbf{P}(d\omega).$$

The distribution function of the expected value of a fuzzy random variable is no longer dependent on the random parameter and therefore is fuzzy.

We will use triangular norms (t-norms) to aggregate fuzzy information. These norms generalize "min" operation inherent in operations on fuzzy sets and fuzzy variables [13,14]. In the work we will use one of the extremal t-norms: $T_M(x,y) = \min\{x,y\}$.

We will now introduce according to [15] the notion of t-relatedness of fuzzy variables. It is used as a tool for building joint possibilities distribution functions.

Definition 3. Fuzzy sets $A_1, \dots, A_n \in P(\Gamma)$ are called mutually T-related if for any index set $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$, $k = 1, \dots, n$, we have

$$\pi(A_{i_1} \cap \dots \cap A_{i_k}) = T\left(\pi(A_{i_1}), \dots, \pi(A_{i_k})\right),$$

where

$$T(\pi(A_{i_1}), \dots, \pi(A_{i_k})) = \mathrm{T}(\mathrm{T}(\dots\mathrm{T}(\pi(A_{i_1}), \pi(A_{i_2})), \pi(A_{i_3})), \dots), \pi(A_{i_k})).$$

We can transfer the concept of mutual T-relatedness of fuzzy sets to fuzzy variables.

Definition 4. Fuzzy variables $Z_1(\gamma), \dots, Z_n(\gamma)$ are called mutually T-related if for any index set $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$, $k = 1, \dots, n$, we have

$$\mu_{Z_{i_1},\dots,Z_{i_k}}(t_{i_1}, \dots, t_{i_k}) = \pi\{\gamma \in \Gamma: Z_{i_1}(\gamma) = t_{i_1}, \dots, Z_{i_k}(\gamma) = t_{i_k}\} = \pi\left\{Z_{i_1}^{-1}\{t_{i_1}\} \cap \dots \cap Z_{i_k}^{-1}\{t_{i_k}\}\right\}$$
$$= T\{\pi(Z_{i_1}^{-1}\{t_{i_1}\}), \dots, \pi(Z_{i_k}^{-1}\{t_{i_k}\})\}, t_{i_j} \in \mathbb{E}^1.$$

3. Construction of a possibilistic-probabilistic programming model under the constraints of possibility/necessity-probability and probability-possibility/necessity

Let's consider a model of possibilistic-probabilistic programming of the following form:

$$k \to \min,$$
$$\tau\{\mathbf{P}\{f_0(x,\omega,\gamma) \le k\} \ge p_0\} \ge \alpha_0, \quad (1)$$
$$\begin{cases} \tau\{\mathbf{P}\{f_i(x,\omega,\gamma) \le 0\} \ge p_i\} \ge \alpha_i, i = 1, \dots, m, \\ x \in X. \end{cases} \quad (2)$$

The probabilistic-possibilistic model associated with (1)-(2) can be written in the form

$$k \to \min,$$
$$\mathbf{P}\{\tau\{f_0(x,\omega,\gamma) \le k\} \ge \alpha_0\} \ge p_0, \quad (3)$$
$$\begin{cases} \mathbf{P}\{\tau\{f_i(x,\omega,\gamma) \le 0\} \ge \alpha_i\} \ge p_i, i = 1, \dots, m, \\ x \in X. \end{cases} \quad (4)$$

It is not difficult to see that the introduced optimization models differ only in the sequence of applying principles to remove hybrid (possibility-probability) uncertainty.

In these models $f_i(x,\omega,\gamma), i = 0, \dots, m$ are fuzzy random functions that have the meaning of mappings

$$f_i(\cdot,\cdot,\cdot): X \times \Omega \times \Gamma \to \mathbb{E}^1,$$

$X \subset \mathbb{E}_n^+ = \{x \in \mathbb{E}^n: x \ge 0\}, \tau \in \{\pi, \nu\}$, $p_i, \alpha_i, i = 0, \dots, m$ are given probability and possibility levels, $p_i, \alpha_i \in (0,1]$, k is an additional scalar variable.

Let's consider specific forms of $f_i(x,\omega,\gamma)$ – linear possibilistic-probabilistic functions. In this case

$$f_i(x,\omega,\gamma)=\sum_{j=1}^{n}A_{ij}(\omega,\gamma)x_j-B_i(\omega,\gamma) \tag{5}$$

with $i=0,\dots,m$ and $B_0(\omega,\gamma)\equiv 0$.

We will assume that $A_{ij}(\omega,\gamma)$ and $B_i(\omega,\gamma)$ have a shift-scale representation of the hybrid (combined) type:

$$A_{ij}(\omega,\gamma)=a_{ij}(\omega)+\sigma_{ij}(\omega)Y_{ij}(\gamma),\qquad B_i(\omega,\gamma)=b_i(\omega)+\sigma_i(\omega)Y_i(\gamma),$$

possibilistic variables $Y_{ij}(\gamma)$, $Y_i(\gamma)$ are characterized by quasiconcave, semi-continuous from above possibilistic distributions with finite support, and $a_{ij}(\omega)\in\mathcal{N}_p(a_{ij}^0,d_{ij}^a)$, $\sigma_{ij}(\omega)\in\mathcal{N}_p(\sigma_{ij}^0,d_{ij}^\sigma)$, $b_i(\omega)\in\mathcal{N}_p(b_i^0,d_i^b)$, $\sigma_i(\omega)\in\mathcal{N}_p(\sigma_i^0,d_i^\sigma)$, $\mathcal{N}_p$ is a class of normal probability distributions.

We introduce notation for the covariance coefficients of random parameters involved in the shift-scale representation of fuzzy random variables:

$$C_{a_{ij}a_{ik}}=cov\left(a_{ij},a_{ik}\right);\quad C_{\sigma_{ij}\sigma_{ik}}=cov\left(\sigma_{ij},\sigma_{ik}\right);$$
$$C_{a_{ij}\sigma_{ik}}=cov\left(a_{ij},\sigma_{ik}\right);\quad C_{\sigma_{ij}a_{ik}}=cov\left(\sigma_{ij},a_{ik}\right);$$
$$C_{b_i\sigma_i}=cov(b_i,\sigma_i);\quad C_{a_{ij}b_i}=cov\left(a_{ij},b_i\right);$$
$$C_{a_{ij}\sigma_i}=cov\left(a_{ij},\sigma_i\right);\quad C_{\sigma_{ij}b_i}=cov\left(\sigma_{ij},b_i\right);$$
$$C_{\sigma_{ij}\sigma_i}=cov\left(\sigma_{ij},\sigma_i\right),$$

and also the covariance matrix $\mathbb{C}^i=\left\{\mathbb{C}_{ijk}\left(t_{ik},t_{ij}\right)\right\}_{j,k=1}^{n}$ with coefficients

$$\mathbb{C}_{ijk}\left(t_{ik},t_{ij}\right)=C_{a_{ij}a_{ik}}+C_{a_{ij}\sigma_{ik}}t_{ik}+C_{\sigma_{ij}a_{ik}}t_{ij}+C_{\sigma_{ij}\sigma_{ik}}t_{ij}t_{ik}.$$

Let the interaction of the fuzzy parameters of the model be described by the strongest t-norm T_M and $t^i=\ (t_{i1},t_{i2},\dots,t_{in},t_i)$ are their fixed values. Then we can write

$$f_i(x,\omega,t^i)=\sum_{j=1}^{n}\left(a_{ij}(\omega)+\sigma_{ij}(\omega)t_{ij}\right)x_j-(b_i(\omega)+\sigma_i(\omega)t_i).$$

It is clear that for fixed x and t^i, $i=1,\dots,m$, $f_i(x,\omega,t^i)$ is a normal random variable. Then with the possibility of

$$\mu(t^i)=\ \min\left\{\min_{1\le j\le n}\left\{\mu_{Y_{ij}}\left(t_{ij}\right)\right\},\mu_{Y_i}(t_i)\right\}$$

its mathematical expectation is determined by the formula [10]

$$m_i(x,t^i)=\mathbf{E}\{f_i(x,\omega,t^i)\}=\sum_{j=1}^{n}\left(a_{ij}^0+\sigma_{ij}^0t_{ij}\right)x_j-(b_i^0+\sigma_i^0t_i).$$

For further analysis, we will also need the variance of the function $f_i(x,\omega,t^i)$. According to the classical approach, it can be defined as follows:

$$d_i(x,t^i)=\mathbf{E}\left\{\left(f_i(x,\omega,t^i)-m_i(x,t^i)\right)^2\right\}.$$

We specify this formula. We have after the corresponding substitutions

$$d_i(x,t^i)=\mathbf{E}\left\{\left(\sum_{j=1}^{n}\left(a_{ij}(\omega)+\sigma_{ij}(\omega)t_{ij}-\left(a_{ij}^0+\sigma_{ij}^0t_{ij}\right)\right)x_j-(b_i(\omega)-b_i^0+(\sigma_i(\omega)-\sigma_i^0)t_i)\right)^2\right\}.$$

After the corresponding permutations and using notations introduced earlier, we have

$$d_i(x,t^i)=\langle\mathbb{C}^ix,x\rangle-2\sum_{j=1}^{n}\left(C_{a_{ij}b_i}+C_{a_{ij}\sigma_i}t_i+C_{\sigma_{ij}b_i}t_{ij}+C_{\sigma_{ij}\sigma_i}t_{ij}t_i\right)x_j+$$
$$+d_i^b+d_i^\sigma t_i^2+2C_{\sigma_ib_i}t_i, \tag{6}$$

where $\langle\cdot,\cdot\rangle$ is the operation of scalar multiplication of vectors.

In the case when the random variables are uncorrelated, the resulting formula (6) takes the form

$$d_i(x,t^i)=\sum_{j=1}^{n}\left(d_{ij}^a+d_{ij}^\sigma t_{ij}^2\right)x_j^2+d_i^b+d_i^\sigma t_i^2\,. \tag{7}$$

The function $d_i(x,t^i)$ has the following properties due to the covariance matrix $\mathbb{C}^i$:

- $d_i(x,t^i)$ is a convex function over x for a fixed t^i;
- for any vectors x and t^i, the function $d_i(x,t^i)$ is non-negative;
- the function $d_i(x,t^i)$ is convex over t^i for a fixed x.

Accordingly, the mathematical expectation of the possibilistic-probabilistic function (5) has the form:

$$\widetilde{m}_i(x,\gamma) = \mathbf{E}\{f_i(x,\omega,\gamma)\} = \sum_{j=1}^{n}\left(a_{ij}^0 + \sigma_{ij}^0 Y_{ij}(\gamma)\right)x_j - \left(b_i^0 + \sigma_i^0 Y_i(\gamma)\right).$$

In the future, we will need the variance formula of the fuzzy random function $f_i(x,\omega,\gamma)$, defined in fuzzy form, following [10]:

$$\tilde{d}_i(x,\gamma) = \mathbf{E}\left\{\left(f_i(x,\omega,\gamma) - \widetilde{m}_i(x,\gamma)\right)^2\right\}.$$

We specify this formula. After substituting specific expressions for $f_i(x,\omega,\gamma)$ и $\widetilde{m}_i(x,\gamma)$ into it, opening the brackets and rearranging the terms, as well as using the linearity property of the mathematical expectation, we obtain the following formula for the fuzzy variance:

$$\begin{aligned}\tilde{d}_i(x,\gamma) = {}&\langle\tilde{\mathbb{C}}^i(\gamma)x,x\rangle\\ &-2\sum_{j=1}^{n}\left(C_{a_{ij}b_i} + C_{a_{ij}\sigma_i}Y_i(\gamma) + C_{\sigma_{ij}b_i}Y_{ij}(\gamma) + C_{\sigma_{ij}\sigma_i}Y_{ij}(\gamma)Y_i(\gamma)\right)x_j + d_i^b + d_i^\sigma Y_i^2(\gamma)\\ &+2C_{\sigma_i b_i}Y_i(\gamma),\end{aligned}$$

where $\tilde{\mathbb{C}}^i(\gamma)$ – is a matrix $\left\{\tilde{\mathbb{C}}_{ijk}(\gamma)\right\}_{j,k=1}^{n}$, $i = 0,\dots,m$, and

$$\tilde{\mathbb{C}}_{ijk}(\gamma) = C_{a_{ij}a_{ik}} + C_{a_{ij}\sigma_{ik}}Y_{ik}(\gamma) + C_{\sigma_{ij}a_{ik}}Y_{ij}(\gamma) + C_{\sigma_{ij}\sigma_{ik}}Y_{ij}(\gamma)Y_{ik}(\gamma).$$

Let us proceed to the construction of equivalent crisp analogs of the problems.

4. Equivalent crisp analogs of problems (1)-(2) and (3)-(4)

First, let's consider the process of constructing an equivalent crisp analog of the problem (1)-(2). To do this, it is enough for us to construct an equivalent deterministic system of constraints (2), after that the model of criterion (1) can be reduced to an equivalent deterministic one in a similar way.

The following results are obtained in [1]. Since for a fixed vector t^i

$$f_i(x,\omega,t^i) \in \mathcal{N}_p\left(m_i(x,t^i),\sqrt{d_i(x,t^i)}\right),$$

then in accordance with the classical results of stochastic programming [16]

$$\begin{aligned}\mathbf{P}\{f_i(x,\omega,t^i) \le 0\} &= \mathbf{P}\left\{\frac{f_i(x,\omega,t^i) - m_i(x,t^i) + m_i(x,t^i)}{\sqrt{d_i(x,t^i)}} \le 0\right\}\\ &= \mathbf{P}\left\{\frac{f_i(x,\omega,t^i) - m_i(x,t^i)}{\sqrt{d_i(x,t^i)}} + \frac{m_i(x,t^i)}{\sqrt{d_i(x,t^i)}} \le 0\right\} = 1 - \mathcal{F}_i\left(\frac{m_i(x,t^i)}{\sqrt{d_i(x,t^i)}}\right) \ge p_i,\end{aligned}$$

where $\mathcal{F}_i$ is a function of the standard normal probability distribution.

The last inequality is equivalent to the inequality $\frac{m_i(x,t^i)}{\sqrt{d_i(x,t^i)}} \le \beta_i$, in which β_i is the solution of the equation $\mathcal{F}_i(t) = 1 - p_i$.

As a result, we get the inequality

$$m_i(x,t^i) - \beta_i\sqrt{d_i(x,t^i)} \le 0 \tag{8}$$

with the possibility of $\mu(t^i)$ equivalent to the i-th constraint of the system (2).

After the specification (8) takes the form

$$\sum_{j=1}^{n}\left(a_{ij}^0 + \sigma_{ij}^0 t_{ij}\right)x_j - \beta_i\sqrt{d_i(x,t^i)} \le b_i^0 + \sigma_i^0 t_i. \tag{9}$$

If $p_i > 0.5$, then $\beta_i < 0$ and the function located on the left side of the inequality (9) is convex and monotonic in fuzzy parameters.

Substituting the corresponding fuzzy values $Y_{ij}(\gamma)$ and $Y_i(\gamma)$ in inequality (9) instead of the parameters t_{ij}, t^i, t_i and requiring the execution of the resulting probability inequality with the possibility α_i, we thereby obtain a constraint equivalent to the i-th constraint of the system (2), containing only fuzzy parameters. We have

$$\pi\left\{\sum_{j=1}^{n}\left(a_{ij}^0 + \sigma_{ij}^0 Y_{ij}(\gamma)\right)x_j - \beta_i\sqrt{\tilde{d}_i(x,\gamma)} \le b_i^0 + \sigma_i^0 Y_i(\gamma)\right\} \ge \alpha_i. \tag{10}$$

In [1], based on the results of [12,17], the following theorem is proved.

Theorem 1. *Let in the constraint model (2)* $\tau = '\pi'$, $p_i > 0.5$, $i = 1, \dots, m$, *the fuzzy variables* $Y_{ij}(\gamma)$, $Y_i(\gamma)$ *in the shift-scale representation of fuzzy random variables are mutually min-related and are characterized by quasi-concave, semi-continuous from above distributions with finite carriers, and the shift and scale coefficients are normally distributed random variables. Then the constraint system (2) is equivalent to a deterministic constraint model*

$$\begin{cases} \sum_{j=1}^{n} \left(a_{ij}^0 + \min\{\sigma_{ij}^0 Y_{ij}^-, \sigma_{ij}^0 Y_{ij}^+\}\right) x_j - \beta_i \sqrt{d_i(x, \alpha_i)} \le b_i^0 + \max\{\sigma_i^0 Y_i^-, \sigma_i^0 Y_i^+\} \\ x \in X, i = 1, \dots, m, \end{cases}$$

in which $Y_{ij}^{\pm}, Y_i^{\pm}$ are the right and left boundaries of α_i level sets of fuzzy variables $Y_{ij}(\gamma)$, $Y_i(\gamma)$,

$$d_i(x, \alpha_i) = \sum_{j=1}^{n} \sum_{k=1}^{n} C_{ijk}(\alpha_i) x_j x_k +$$

$$+ 2 \sum_{j=1}^{n} \Big(-C_{a_{ij} b_i} + \min\{-C_{a_{ij}\sigma_i} Y_i^-, -C_{a_{ij}\sigma_i} Y_i^+\} + \min\{-C_{\sigma_{ij} b_i} Y_{ij}^-, -C_{\sigma_{ij} b_i} Y_{ij}^+\}$$

$$+ \min\{-C_{\sigma_{ij}\sigma_i} (Y_{ij} Y_i)^-, -C_{\sigma_{ij}\sigma_i} (Y_{ij} Y_i)^+\} \Big) x_j + d_i^b + d_i^\sigma (Y_i^2)^- + 2\min\{C_{\sigma_i b_i} Y_i^-, C_{\sigma_i b_i} Y_i^+\},$$

$$C_{ijk}(\alpha_i) = C_{a_{ij} a_{ik}} + \min\left\{C_{a_{ij}\sigma_{ik}} Y_{ik}^-, C_{a_{ij}\sigma_{ik}} Y_{ik}^+\right\} + \min\left\{C_{\sigma_{ij} a_{ik}} Y_{ij}^-, C_{\sigma_{ij} a_{ik}} Y_{ij}^+\right\}$$

$$+ \min\left\{C_{\sigma_{ij}\sigma_{ik}} (Y_{ij} Y_{ik})^-, C_{\sigma_{ij}\sigma_{ik}} (Y_{ij} Y_{ik})^+\right\}.$$

Remark 1. The boundaries of α_i-level sets of fuzzy quantities, their powers and products are found using the results presented in [10,18,19,20]. Eg:

$$(Y_i Y_{ij})^- = \min\{Y_i^- Y_{ij}^-, Y_i^- Y_{ij}^+, Y_i^+ Y_{ij}^-, Y_i^+ Y_{ij}^+\},$$
$$(Y_i Y_{ij})^+ = \max\{Y_i^- Y_{ij}^-, Y_i^- Y_{ij}^+, Y_i^+ Y_{ij}^-, Y_i^+ Y_{ij}^+\}.$$

In [1], a theorem is also presented that allows us to construct an equivalent crisp analog of the criterion model (1).

Theorem 2. *Let in the criterion model (1)* $\tau = '\pi'$, $p_0 > 0.5$, $i = 1, \dots, m$, *the fuzzy variables* $Y_{0j}(\gamma)$ *in the shift-scale representation of fuzzy random variables are mutually min-related and are characterized by quasi-concave, semi-continuous from above distributions with finite carriers, and the shift and scale coefficients are normally distributed random variables. Then the model of criterion (1) is equivalent to the model*

$$\sum_{j=1}^{n} \left(a_{0j}^0 + \min\{\sigma_{0j}^0 Y_{0j}^-, \sigma_{0j}^0 Y_{0j}^+\}\right) x_j - \beta_0 \sqrt{\sum_{j=1}^{n} \sum_{k=1}^{n} C_{0jk}(\alpha_0) x_j x_k} \to \min_{x \in X}.$$

The objective function in the last problem is convex, and in the case of $p_0 < 0.5$, the problem goes into the class of non-convex multiextremal optimization. Nevertheless, the condition $p_0 > 0.5$ meets the requirements of most problems solved in practice.

Let's proceed to the study of the model (3)-(4). We denote by $\mathcal{N}_\pi(\hat{a}, \hat{b})$ the class of normal possibilistic variables:

$$\chi \in \mathcal{N}_\pi(\hat{a}, \hat{b}) \Rightarrow \mu_\chi(x) = e^{-\frac{(x - \hat{a})^2}{\hat{b}^2}} \quad \forall x \in \mathbb{E}^1.$$

Theorem 3. *Let in the constraint model (4) the possibilistic components in the shift-scale representation of fuzzy random variables* $A_{ij}(\omega, \gamma)$ *and* $B_i(\omega, \gamma)$ *be normal* T_M*-related possibilistic variables:*

$$Y_{ij}(\gamma) \in \mathcal{N}_\pi(\hat{a}_{ij}, \hat{d}_{ij}), \;\; Y_i(\gamma) \in \mathcal{N}_\pi(\hat{b}_i, \hat{d}_i).$$

Then the possibilistic-probabilistic model of constraints (4) has an equivalent stochastic analog of the form

$$\begin{cases} \mathbf{P}\left\{\sum_{j=1}^{n} A_{ij}^{\{\mp\}}(\omega) x_j \le B_i^{\{\pm\}}(\omega)\right\} \ge p_i, i = 1, \dots, m, \\ x \in X, \end{cases} \tag{11}$$

where

$$A_{ij}^-(\omega) = a_{ij}(\omega) + \hat{a}_{ij}\sigma_{ij}(\omega) - \hat{\sigma}_{ij} |\sigma_{ij}(\omega)| \sqrt{-\ln \alpha_i},$$

$$B_i^+(\omega) = b_i(\omega) + \hat{b}_i\sigma_i(\omega) + \hat{\sigma}_i|\sigma_i(\omega)|\sqrt{-\ln\alpha_i},$$
$$A_{ij}^+(\omega) = a_{ij}(\omega) + \hat{a}_{ij}\sigma_{ij}(\omega) + \hat{\sigma}_{ij}|\sigma_{ij}(\omega)|\sqrt{-\ln\beta_i},$$
$$B_i^-(\omega) = b_i(\omega) + \hat{b}_i\sigma_i(\omega) - \hat{\sigma}_i|\sigma_i(\omega)|\sqrt{-\ln\beta_i},$$

$\beta_i = 1 - \alpha_i$; *the upper indices (- +) in the notation of the boundaries of level sets of possibilistic variables correspond to* $\tau = \pi$ *(possibility constraint), the lower indices (+ -) correspond to* $\tau = \nu$ *(necessity constraint).*

Proof. Let $\tau = \pi$. For a fixed $\omega \in \Omega$, in accordance with the calculus of possibilities [10]

$$A_{ij}(\omega,\gamma) = a_{ij}(\omega) + \sigma_{ij}(\omega)Y_{ij}(\gamma) \in \mathcal{N}_\pi\big(a_{ij}(\omega) + \hat{a}_{ij}\sigma_{ij}(\omega), |\sigma_{ij}(\omega)|\hat{d}_{ij}\big)$$
$$B_i(\omega,\gamma) = b_i(\omega) + \sigma_i(\omega)Y_i(\gamma) \in \mathcal{N}_\pi\big(b_i(\omega) + \hat{b}_i\sigma_i(\omega), |\sigma_i(\omega)|\hat{d}_i\big).$$

Then, based on the results of [21], the possibilistic restriction

$$\pi\left\{\sum_{j=1}^{n} A_{ij}(\omega,\gamma)x_j \le B_i(\omega,\gamma)\right\} \ge \alpha_i$$

has an equivalent stochastic analog

$$\sum_{j=1}^{n} A_{ij}^-(\omega)x_j \le B_i^+(\omega).$$

It follows from the resulting inequality that the probabilistic/possibilistic constraint

$$\mathbf{P}\left\{\pi\left\{\sum_{j=1}^{n} A_{ij}(\omega,\gamma)x_j \le B_i(\omega,\gamma)\right\} \ge \alpha_i\right\} \ge p_i$$

is equivalent to the probabilistic constraint

$$\mathbf{P}\left\{\sum_{j=1}^{n} A_{ij}^-(\omega)x_j \le B_i^+(\omega)\right\} \ge p_i.$$

Let now $\tau = \nu$. The proof in this case is based on the relation of the duality of the possibility and necessity measures [10]:

$$\nu(A) = 1 - \pi(A^C), \qquad \forall A \in P(\Gamma)$$

and is proved in the same way. The theorem is proved.

Theorem 3 can be generalized on the basis of the results of [17] to the class of quasi-concave semi-continuous from above distributions.

The resulting system of constraints (11) is a system of row-by-row constraints on probability. The construction of its equivalent deterministic analogue is difficult due to the presence of absolute values of random variables in the expressions for random variables $A_{ij}^{\{\mp\}}(\omega)$ and $B_i^{\{\pm\}}(\omega)$. So, for example, if the random variable is $\xi \in \mathcal{N}_p(0,1)$, then the probability density function f of the random variable $|\xi|$ has the form [22]:

$$f_{|\xi|}(y) = \begin{cases} f_\xi(y) + f_\xi(-y), y > 0, \\ 0, y < 0. \end{cases}$$

However, with the help of characteristic functions

$$\chi^i(x,\omega) = \begin{cases} 1, f^i(x,\omega) \le 0, \\ 0, f^i(x,\omega) > 0, \end{cases}$$

where $f^i(x,\omega) = \sum_{j=1}^{n} A_{ij}^{\{\mp\}}(\omega)x_j - B_i^{\{\pm\}}(\omega)$ this system can be reduced to a system of restrictions

$$\begin{cases} \mathbf{E}\{\chi^i(x,\omega)\} = \mathbf{P}\{f^i(x,\omega) \le 0\} \ge p_i, i = 1,\dots,m, \\ x \in X. \end{cases} \tag{12}$$

After that we can apply direct stochastic programming methods [16], which do not require the construction of equivalent deterministic analogues.

An equivalent stochastic model of criterion (3) can be constructed in a similar way.

5. Minimum risk portfolio models

Based on the obtained results, we now can construct a mathematical model of a minimal risk portfolio under conditions of hybrid uncertainty, in which possibility-probability constraints are used.

Indeed, let the possibilistic-probabilistic function

$$f_0(x,\omega,\gamma) = \sum_{j=1}^{n} A_{0j}(\omega,\gamma)x_j,$$

in which $A_{0j}(\omega,\gamma)$ is the return of the j-th financial asset, and the vector $x = (x_1, \dots, x_n)$ is the vector of equity shares, represents the return of the investment portfolio.

Then one of the possible models of the minimum risk portfolio can be written as

$$k \to min,$$

$$\tau\{\tilde{\mathrm{d}}_0(x,\gamma) \leq k\} \geq \alpha_0, \quad (13)$$

$$\begin{cases} \tau\{\mathbf{P}\{f_0(x,\omega,\gamma) \geq d_p\} \geq p_0\} \geq \alpha_0, \\ \sum_{j=1}^{n} x_j = 1, x \geq 0, \end{cases} \quad (14)$$

where d_p is the level of profitability acceptable to an investor. The system (13)-(14) is a possibilistic-probabilistic programming model. Under the assumptions made earlier, its equivalent deterministic analogue – the quadratic programming problem – can be constructed.

6. Conclusion

The paper investigates the problem of possibilistic-probabilistic linear programming with restrictions on the possibility/necessity - probability and probability - possibility/necessity. For the case of normally distributed random factors of the model, with the most general assumptions about the properties of probability distributions, its equivalent deterministic and stochastic analogues are constructed. The properties of the model (convexity-concavity), as in classical stochastic programming [16], significantly depend on the probability levels set in the original model. For the case of the measure of necessity ($\tau = \nu$) in the equivalent model (we can prove this by relying on [12]), the right boundaries of the level sets for $1 - \alpha_i$ will be used in the left part of the inequalities included in the system (11), and the left ones in the right part.

It can be shown that for probability values $p_i = 0.5$, second-order moments (variances) are excluded and the model (1)-(2) is transformed into a linear programming problem. The model of the minimum risk portfolio is formulated. It can be investigated in the context of the obtained results of probabilistic optimization.

In terms of prospective studies, it seems appropriate to generalize the results obtained to probability distributions other than normal, as well as to the case of the weakest t-norm describing the interaction of fuzzy parameters [23,24].

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8. References

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