Use autoregressions with multiple roots of the characteristic equations to image representation and filtering

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Abstract. The article describes probabilistic properties of autoregressive models with multiple roots of characteristic equations as well as the results of such properties investigation. Particular attention is paid to the study of covariance functions. However, we investigate only such models of different orders that provides equal intervals of correlation in row and column. So the multiplicity of such models by row and column is the same. Asymptotic graphs of covariance functions cross-sections are constructed. It is shown that the cross-sections tend to ellipses when model orders are improving. We also describe in detail the problem of filtering images generated by autoregressions with multiple roots. Dependences of variances of filtering errors on the correlation parameter are obtained. The efficiency of filtering at various signal noise ratios and various orders of models is investigated. The effectiveness and expediency of applying autoregressions with multiple roots are shown in comparison with other autoregressions of high orders.

1. Introduction

A lot of real images is characterized by a smooth change in brightness. It means that real images have rather strong correlation links in some neighborhood. The known autoregressive (AR) image models [1-3] either do not adequately describe the nature of such images even at high values of the correlation coefficient in case of first order model.

However, the use of higher order models requires complex operations to calculate the set of correlation parameters. Meanwhile, there are models of AR random field (RF), that generated by the characteristic equations in image processing and representation literature [4-6,12]. Such models are called AR with multiple roots. In this case, the application of algorithms based on such models, when processing real signals and in various applied problems, can contribute to improving the efficiency of solving such problems. In recent years, particular interest is caused by the processing of satellite images [13-15].

A lot of tasks connecting with preliminary processing are known, but the important task is to suppress noise or filter images [7,8]. The advantage of models with multiple roots is the possibility of applying recurrent filtration procedures [9]. The article deals with the problem of image modeling on the basis of such models as well as problems of analysis of probability properties and optimal filtering of random fields generated by such models.

2. Autoregressions with multiple roots of characteristic equations

The widely known class of image models is the AR model of RF [1-3]. One of the main reasons for the spread of AR models is the efficient mathematical apparatus developed for modeling random sequences. The AR class of RF models can be obtained on the basis of linear stochastic difference equations of the following form:

$$x_i = \sum_{\bar{j} \in D} \alpha_{\bar{j}} x_{\bar{i}-\bar{j}} + \beta \xi_i \quad \bar{i} \in \Omega, \tag{1}$$

where $X = \{x_{\bar{i}}, \bar{i} \in \Omega\}$ is modeled RF defined on the N-size grid $\Omega = \{\bar{i} = (i_1, i_2, ..., i_N): \ \{i_k = \overline{1..M_k}\}, \ k = \overline{1..N}\}$; $\{\alpha_{\bar{j}}, \beta, \bar{j} \in D\}$ are coefficients of the model; $\{\xi_{\bar{i}}, \bar{i} \in \Omega\}$ is a random variable with Gaussian distribution; $D \subset \Omega$ is a causal region of local states.

The choice of a normally distributed RF with independent components is convenient and simple enough to describe some brightness properties of the image by the model. In this case, the RF X will also obey the Gaussian distribution. As an example, let us consider the formation of a two-dimensional RF using of the first order AR model known also as Habibie model:

$$x_{i,j} = \rho_x x_{i-1,j} + \rho_y x_{i,j-1} - \rho_x \rho_y x_{i-1,j-1} + \xi_{i,j}, i = \overline{1..M_1}; \ j = \overline{1..M_2}, \tag{2}$$

where ρ_x and ρ_y are coefficients of correlation of neighboring elements in row and column, respectively; $\{\xi_{i,j}\}$ is a two-dimensional field of independent Gaussian random variables with zero mean $M\{\xi_{i,j}\} = 0$ and variance $\sigma_\xi^2 = M\{\xi_{i,j}^2\} = (1-\rho_x^2)(1-\rho_y^2)\sigma_x^2$; $\sigma_x^2 = M\{x_{i,j}^2\}$; $M_1 \times M_2$ *is* the size of the image that we imitate.

Due to the specifics of this way to generate the image, the process of estimating the parameters of the AR model can be performed without special difficulties. Nevertheless, such models are not good enough for describing real images, since they provide only small regions of local states. And this is an important factor in modeling multi-zone images. Therefore, if it is necessary to use the AR equation, we must increase the order of the model for the expansion of local states. So this fact entails an inevitable growth of computational costs. Sometimes the image can be so difficult that it is required to perform imitation in a class other than AR. Otherwise, to achieve adequate replacement of the image is unlikely to succeed.

In addition, the weakness of such models is that they are not suitable for an adequate description of isotropic RFs, for example, with a covariation function (CF) of the form $R(\bar{k}) = R(\bar{k} = \sqrt{k_1^2 + k_2^2 + ... + k_N^2})$. To overcome this drawback, one can use the characteristic equations [10]. In work [6] models based on the generalization of one-dimensional ARs to the multidimensional case were proposed. For example, if we take an AR with multiple roots of characteristic equations of the second order

$$x_i = 2\rho x_{i-1} - \rho^2 x_{i-2} + \xi_i, \tag{3}$$

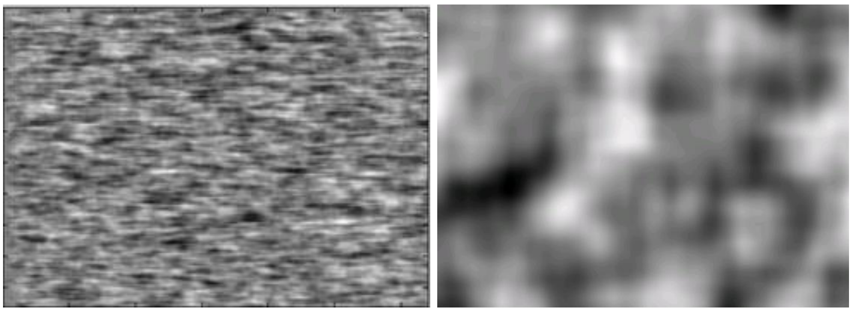
then it is possible to obtain on its basis a model of a two-dimensional second-order RF

$$\begin{aligned} x_{ij} &= 2\rho_x x_{i-1,j} + 2\rho_y x_{i,j-1} - 4\rho_x \rho_y x_{i-1,j-1} - \\ &- \rho_x^2 x_{i-2,j} - \rho_y^2 x_{i,j-2} + 2\rho_x^2 \rho_y x_{i-2,j-1} + \\ &+ 2\rho_y^2 \rho_x x_{i-1,j-2} - \rho_x^2 \rho_y^2 x_{i-2,j-2} + b\xi_{ij}, \end{aligned} \tag{4}$$

where b is normalizing coefficient, which makes it possible to obtain a stationary RF with a given variance.

It can be noted that the model (4) is an eight-point model, i.e in it to form the next element of the RF $\{x\}$ we use 8 preceding elements from the neighborhood. Similarly, for the multiplicity model (3.3), we can obtain a 15-point model, for the multiplicity model (4.4), we obtain a 24-point model.

Figure 1(a) shows an image formed from the values of RF (4) with model parameters $\rho_x = 0.9$ and $\rho_y = 0.7$. Figure 1(b) shows the implementation of an RF based on AR with multiple roots of characteristic equations having model parameters $\rho_x = 0.95$ and $\rho_y = 0.95$ with multiplicity (3,3). The size of images is *640×480* pixels.



(a) multiplicity (2;2) (b) multiplicity (3;3)

Figure 1. Implementation of models having different multiplicities.

Thus, using a model with multiple roots of characteristic equations when imitating images, it is possible to simply obtain such implementations of RF that will be close to real images. In this case, an important property of the generated RF will be their quasi-isotropy property. A general formula for models of different multiplicities can be written in the form:

$$x_{i,j} = \beta\xi_{i,j} - \sum_{i_1=0}^{N_1}\sum_{j_1=0}^{N_2}\alpha_{i_1,j_1}x_{i-i_1,j-j_1}, \qquad (5)$$

where N_1 and N_2 characterize the multiplicity of the model; coefficients $\alpha_{i_1,j_1}(\alpha_{0,0}=0)$ are products of the corresponding coefficients of one-dimensional AR along the axes x and y:

$$\alpha_{i_1,j_1} = \alpha x_{i_1}\alpha y_{j_1}. \qquad (6)$$

The coefficients of one-dimensional AP (6) can be obtained from the expression

$$\begin{aligned}\alpha x_{i_1}(\rho_x, N_1) &= (-1)^{i_1+1}C_{N_1}^{i_1}\rho_x^{i_1}\\ \alpha y_{j_1}(\rho_y, N_2) &= (-1)^{j_1+1}C_{N_2}^{j_1}\rho_x^{j_1}\end{aligned} \qquad (7)$$

where $C_n^m = \dfrac{n!}{m!(n-m)!}$ is number of combinations of n by m; ρ_x, ρ_y are parameters of the model.

Finally, the two-dimensional model coefficient β is the normalized product of the corresponding coefficients of one-dimensional ARs along the axes x and y:

$$\beta = \frac{\sigma_x}{\sigma_\xi}\beta_x\beta_y. \qquad (8)$$

These coefficients can be found from the following formulas

$$\beta_x = \sqrt{\frac{\left(1-\rho_x^2\right)^{2N_1-1}}{\sum_{i=0}^{N_1-1}\left(C_{N_1-1}^{i}\rho_x^{i}\right)^2}}, \beta_y = \sqrt{\frac{\left(1-\rho_y^2\right)^{2N_2-1}}{\sum_{i=0}^{N_2-1}\left(C_{N_2-1}^{i}\rho_y^{i}\right)^2}}. \qquad (9)$$

So we can use expressions (5) - (9) to show, for example, that the AR model of the first order (2) can be represented by a model with multiple roots of multiplicity (1,1), and the eight-point model (4) can be represented by a model with multiple roots with multiplicities (2,2).

Thus, it is possible to construct models of arbitrary multiplicity, obtaining images with slow varying brightness properties. It should be noted that speed of properties change will depend on parameters and orders of models.

3. Covariance functions of autoregressive random fields with multiple roots

In order to construct AR models of images with multiple roots of characteristic equations having given statistical characteristics, we perform an analysis of the data of the characteristics of the considered models. An important task arising in the course of statistical analysis of the mathematical model of an RF is to determine the type of its CF.

An important property of models with multiple roots is the factorizability of CF. For example, the RF generated by the model (1) is anisotropic, and its covariance CF, by virtue of anisotropy, is a generalization of the CF of a one-dimensional first order AR to a two-dimensional case. It can be shown [4] that it is described by the following expression:

$$B(k_1,k_2)=\sigma_x^2\rho_x^{|k_1|}\rho_y^{|k_2|}, \tag{10}$$

where σ_x^2 is the variance of the RF X; ρ_x and ρ_y are parameters of the model; k_1 and k_2 are the distances between the elements of the RF X along the axes *x* and *y*.

The use of the model (4) provides an increase in the links in the AR model due to the expansion of the range of significant preceding states and the type of CF changes. For model (4), the CF significantly differs from the CF of the first order AR model and it takes the following form [4]:

$$B(k_1,k_2)=\sigma_x^2(1+\frac{1-\rho_x^2}{1+\rho_x^2}|k_1|)(1+\frac{1-\rho_y^2}{1+\rho_y^2}|k_2|)\rho_x^{|k1|}\rho_y^{|k2|}, \tag{11}$$

where σ_x^2 is variance of the RF X; ρ_x, ρ_y are correlation parameters of the model; k_1 and k_2 are the distances between the elements of the RF X along the axes *x* and *y*.

In order to obtain the CF of models with arbitrary orders, one can use expressions for one-dimensional CF of AR with multiple roots of the characteristic equations

$$B_x(k)=\sigma_x^2\sum_{l=0}^{m-1}g(m,l,k)\frac{\rho^{2(m-l-1)}}{(1-\rho^2)^{2k-l-1}}, \tag{12}$$

where $g(m,l,k)=\dfrac{(m+k-1)!(2m-l-2)!}{l!(m-1)!(m-l-1)!(m+k-l-1)!}$. The variance of RF ξ_i, $i=1,2,...,n$ can be found by using condition $B_x(0)=\sigma_x^2$

$$\sigma_\xi^2=\sigma_x^2(1-\rho^2)^{2m-1}\Big/\sum_{l=0}^{m-1}(C_{m-1}^l\rho^l)^2. \tag{13}$$

Correspondingly, if the AR multiplicity is *(m₁,m₂)* then the expression for the CF can be written as the product of the RF variance and expressions of the form (12) for one-dimensional CF

$$B_{xy}(k_1,k_2)=\sigma_x^2\sum_{l=0}^{m_1-1}g(m_1,l,k_1)\frac{\rho_x^{2(m_1-l-1)}}{(1-\rho_x^{\,2})^{2k_1-l-1}}\sum_{l=0}^{m_2-1}g(m_2,l,k_2)\frac{\rho_y^{2(m_2-l-1)}}{(1-\rho_y^{\,2})^{2k_2-l-1}}. \tag{14}$$

The obtained relations completely determine the CF and the parameters of the RFs generated by the AR with multiple roots of the characteristic equations of multiplicity *(m_1,m_2)*.

For analysis, an interesting case is when the correlation coefficients of the model provide the same correlation intervals on row and column for models of different orders, i.e.

$$B_{m=1}(\rho_{x1},\rho_{y2},k_0,k_0)=B_{m=2}(\rho_{x2},\rho_{y2},k_0,k_0)=...=B_{m=n}(\rho_{xn},\rho_{yn},k_0,k_0)=...=\frac{\sigma_x^2}{e}. \tag{15}$$

For simplicity, we will assume that the multiplicity of the AR for each of the axes is the same, and the parameter ρ is also the same for both axes. Then we can reduce condition (15) to the following form

$$B_{m=n}(\rho,k_0)=\frac{\sigma_x}{\sqrt{e}}. \tag{16}$$

This simplification makes it possible to obtain a set of parameters that ensure the same correlation intervals, analogous to the one-dimensional case [5]. Table 1 presents the dependences between the correlation parameter ρ and correlation interval k_0.

Analysis of the data presented in Table 1 shows that in order to ensure equal correlation interval, it is necessary to decrease the value of the parameter ρ. Furthermore, it is possible to single out the dependence of expression $\gamma_m=2(1-\rho_m)k_0$. The parameter γ_m tends to 1 if $m=1$. Figure 2 shows dependence of parameter γ_m on correlation interval k_0.

Table 1. Results of numerical calculations of parameters for different correlation intervals.

k_0	1	10	20	50	100	500	1000	10000
$\rho_{m=1}$	0,606	0,9048	0,9512	0,9672	0,9802	0,99004	0,99502	0,9994998
$\rho_{m=2}$	0,338	0,7657	0,8737	0,9137	0,9472	0,9732	0,98653	0,998644
$\rho_{m=3}$	0,2345	0,685	0,8257	0,8797	0,926	0,96225	0,98093	0,998077
$\rho_{m=4}$	0,1795	0,6275	0,7895	0,854	0,9095	0,9536	0,97653	0,99763

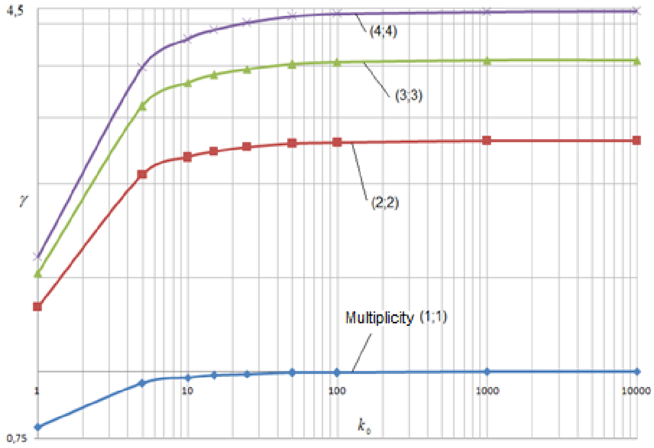
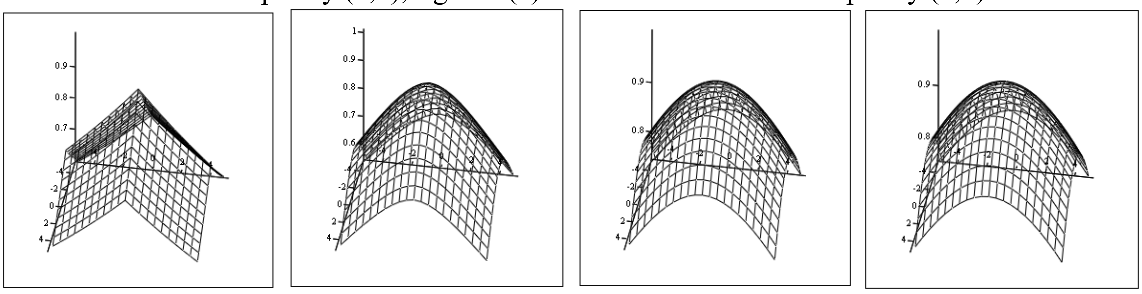


Figure 2. Dependence of the correlation characteristics for models of different orders.

Analysis of the curves in Figure 2 shows that the parameter γ_m tends to a constant value for large k_0. At the same time, the greater the multiplicity of the model, then the greater the parameter γ_m.

Figure 3 shows CF of different order models providing a correlation interval $k_0 = 10$. Figure 3(a) shows the model of multiplicity (1,1), figure 3(b) shows the model of multiplicity (2,2), Figure 3(c) shows the model of multiplicity (3,3), figure 3(d) shows the model of multiplicity (4,4).



(a) (b) (c) (d)

Figure 3. CF of the AR RF with multiple roots of the characteristic equations.

Figure 4 shows the CF cross-sections for the models of the 1st order (solid line) and the 2nd order (dashed line) with the correlation interval $k_0 = 15$. Figure 5 shows CF cross-sections for models of different orders with the same value of the parameter $\rho = 0.8615$.

As can be seen from Figure 4, the second order CF model has a "bell-shaped" vertex, and the correlation links between the RF elements generated by such models are stronger within the correlation interval.

Fig. 5 shows that the CF cross-sections tend to ellipsoids with increasing multiplicities AR. However, with a significant increase in the orders, the CF decreases much more slowly than when using the first and second order ARs.

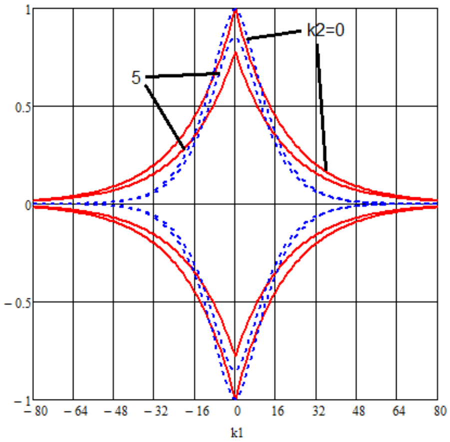


Figure 4. CF cross-sections with the same correlation interval.

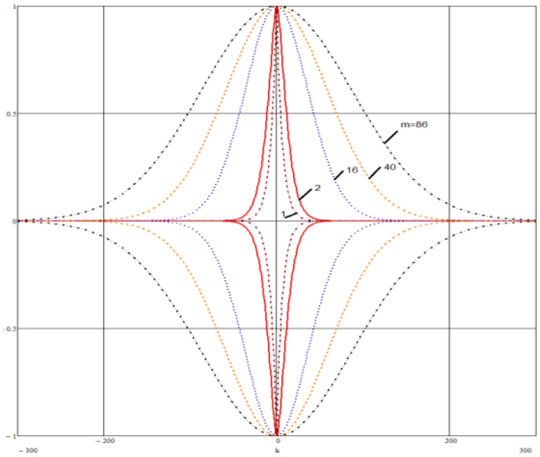


Figure 5. CF cross-sections with the same correlation parameter.

Thus, the correlation properties of AR with multiple roots of the characteristic equations ensuring equal correlation intervals are investigated. Dependences of the correlation parameter of such ARs on the correlation interval are obtained.

4. Covariance functions of autoregressive random fields with multiple roots

Let us consider the case when against a background of white noise, it is necessary to perform a filtration of an RF simulated by the AR with multiple roots of the characteristic equations

$$z_{ij} = x_{ij} + n_{ij}, \tag{17}$$

where white noise has zero mathematical expectation and variance σ_n^2.

We will use the following simple algorithm for image filtering. At the first stage, we perform line by line filtering of all the elements in each row. Then the same procedure applies to each column. Finally, we get the total estimation of each element as the average between the estimates for the row and the column. So, to filter a string, you can use the following algorithm [5]. We introduce the extended state vector:

$$\bar{x}_i = (x_i x_{i-1} \ldots x_{i-m+1})^T. \tag{18}$$

Then the observation model will be written as:

$$z_i = C\bar{x}_i + n_i, \; i = 1, 2, \ldots, \tag{19}$$

where $C = (1\,0 \ldots 0)$.

The equation of state of a string is also can be written in the vector-matrix form:

$$\bar{x}_i = \wp \bar{x}_{i-1} + \bar{\xi}_i, i = 1, 2, \ldots, \tag{20}$$

where $\wp = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1m} \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 1 \end{pmatrix}$; $\rho_{1j} = (-1)^{j+1} C_m^j \rho^j$; $\bar{\xi}_i = (\xi_i 0 \dots 0)^T; V_\xi = M\{\bar{\xi}_i \bar{\xi}_i^T\}$.

After the transformations, we use the standard Kalman linear filter equations to find the optimal estimates of the information RF [11]:

$$\hat{\bar{x}}_i = \hat{\bar{x}}_{эi} + P_i C^T \frac{1}{\sigma_n^2}(z_i - C\hat{\bar{x}}_{эi}), P_i = P_{эi}(E + \frac{1}{\sigma_n^2} C^T C P_{эi})^{-1}, \tag{21}$$

where $P_{эi} = \wp P_{i-1} \wp^T + V_\xi ; \hat{\bar{x}}_{эi} = \wp \hat{\bar{x}}_{i-1}$.

At the every i-th estimation step we build the optimal forecast $\hat{x}_{эi} = \sum_{j-1}^{m} \rho_{1j} \hat{x}_{i-j}$ based on previous estimates $\hat{x}_{i-j}$, $j = 1,2,\dots,m$ and we find the best estimate (in the sense of the minimum variance of the error) $P_{11i} = M\left\{(x_i - \hat{x}_i)^2\right\}$

$$\hat{x}_i = \hat{x}_{эi} + P_{11i} \frac{1}{\sigma_n^2}(z_i - \hat{x}_{эi}), \tag{21}$$

where $P_{11i} = P_{э11i} \Big/ \left(1 + P_{э11i} \big/ \sigma_n^2\right)$. The remaining components of the vector $\hat{\bar{x}}_i$ are calculated on the basis of interpolation of previous estimates taking into account the next observation z_i and the mutual covariance of estimation errors.

Figure 6 shows the variances of filtering errors obtained for models of different multiplicity and for different ratios of signal to noise $q = \sigma_x^2 / \sigma_n^2$. It is worth noting that the variance estimate of the filter error was averaged over five processed images of size *150×150*.

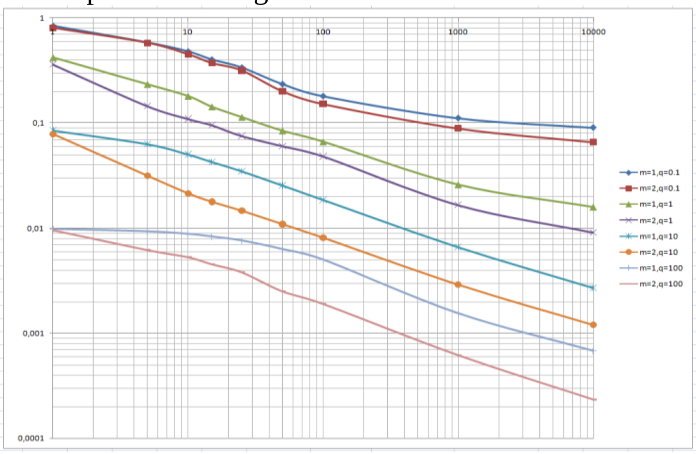


Figure 6. The filtration efficiency of two-dimensional ARs with multiple roots of characteristic equations.

Figure 7 shows the results of the proposed algorithm for the models with multiplicities (1;1) and (2;2). The filtering parameters are following: $q = 1$, $k_0 = 15$, image size is *300×300*. From the top down we show the original images, noisy images, images obtained after filtering.

An analysis of the results shows that the use of AR of higher orders, firstly, makes it possible to obtain smoother fields, and secondly, it provides a greater filtration efficiency (on average 32% for 1-st and 2-nd order models with q=1).

5. Conclusion

Thus, RF models based on AR with multiple roots of characteristic equations are presented. The equality of correlation interval is the interesting property of investigated models. For such models, dependencies between correlation properties and multiplicities are found. An effective quasi-optimal filtering algorithm for such RFs is proposed. It is shown that the use of high-order models makes it possible to obtain gains in the case of filtering in the case of identical correlation intervals.

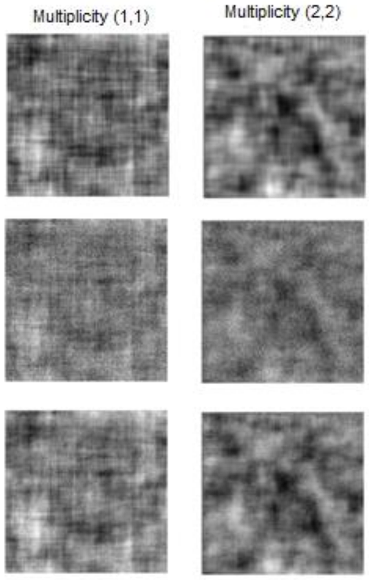


Figure 7. The results of filtering simulated images.

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Acknowledgments
The study was supported by RFBR, project № 17-01-00179.