L-Bonds vs extents of direct product of two L-fuzzy contexts

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Abstract. We focus on the direct product of two L-fuzzy contexts, which are defined with the help of a binary operation on a lattice of truth-values L. This operation, essentially a disjunction, is defined as $k \ltimes l = \neg k \to l$, for $k, l \in L$ where negation is interpreted as $\neg l = l \to 0$. We provide some results which extend previous work by Krötzsch, Hitzler and Zhang.

1 Introduction

Formal concept analysis (FCA) introduced by Ganter and Wille [9] has become an extremely useful theoretical and practical tool for formally describing structural and hierarchical properties of data with "object-attribute" character. Regarding applications, we can find papers ranging from ontology merging [20], to applications to the Semantic Web by using the notion of concept similarity [7], and from processing of medical records in the clinical domain [11] to the development of recommender systems [6].

Soon after the introduction of "classical" formal concept analysis, several approaches towards its generalization were introduced and, nowadays, there are recent works which extend the theory by using ideas from fuzzy set theory, or fuzzy logic reasoning, or from rough set theory, or some integrated approaches such as fuzzy and rough, or rough and domain theory [1, 15–18, 21, 22].

In this paper, we are concerned with extensions of Bělohlávek's approach. In [2, 4] he provided an L-fuzzy extension of the main notions of FCA, such as context and concept, by extending its underlying interpretation on classical logic to the more general framework of L-fuzzy logic [10].

In this work, we aim at formally describing some structural properties of intercontextual relationships [8] of L-fuzzy formal contexts. The categorical treatment of morphisms as fundamental structural properties has been advocated by [14] as a means for the modelling of data translation, communication, and distributed computing, among other applications. Research on (extensions of) the theory of Chu spaces studies morphisms among contexts in order to obtain categories with certain specific properties. Previous work in this line has been

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developed by the authors in [12] by using category theory following the results in [19].

The main result here is the extension of the relationship between bonds and extents of direct products of contexts to the realm of L-fuzzy FCA.

2 Preliminary definitions

In order to make this contribution as self-contained as possible, we proceed now with the preliminary definitions of complete residuated lattice, L-fuzzy context, L-fuzzy concept.

2.1 *L*-fuzzy concept lattice

Definition 1. *An algebra* $\langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ *is said to be a* ***complete residuated lattice*** *if*

1. $\langle L, \wedge, \vee, 0, 1\rangle$ *is a complete bounded lattice with the least element* 0 *and the greatest element* 1*,*
2. $\langle L, \otimes, 1\rangle$ *is a commutative monoid,*
3. $\otimes$ *and* $\rightarrow$ *are adjoint, i.e.* $a \otimes b \leq c$ *if and only if* $a \leq b \rightarrow c$*, for all* $a, b, c \in L$*, where* $\leq$ *is the ordering in the lattice generated from* $\wedge$ *and* $\vee$*.*

Now, the natural extension of the notion of context is given below.

Definition 2. *Let* L *be a complete residuated lattice, an* L***-fuzzy context*** *is a triple* $\langle B, A, r\rangle$ *consisting of a set of objects* B*, a set of attributes* A *and an* L*-fuzzy binary relation* r*, i.e. a mapping* $r \colon B \times A \rightarrow L$*, which can be alternatively understood as an* L*-fuzzy subset of* $B \times A$*.*

We now introduce the L-fuzzy extension provided by Bělohlávek [2,3], where we will use the notation Y^X to refer to the set of mappings from X to Y.

Definition 3. *Given an* L*-fuzzy context* $\langle B, A, r\rangle$*, a pair of mappings* $\uparrow \colon L^B \rightarrow L^A$ *and* $\downarrow \colon L^A \rightarrow L^B$ *can be defined for every* $f \in L^B$ *and* $g \in L^A$ *as follows:*

$$\uparrow f(a) = \bigwedge_{o \in B} (f(o) \rightarrow r(o, a)) \qquad\qquad \downarrow g(o) = \bigwedge_{a \in A} \big(g(a) \rightarrow r(o, a)\big) \tag{1}$$

Lemma 1. *Let* L *be a complete residuated lattice, let* $r \in L^{B \times A}$ *be an* L*-fuzzy relation between* B *and* A*. Then the pair of operators* $\uparrow$ *and* $\downarrow$ *form a Galois connection between* $\langle L^B; \subseteq\rangle$ *and* $\langle L^A; \subseteq\rangle$*, that is,* $\uparrow \colon L^B \rightarrow L^A$ *and* $\downarrow \colon L^A \rightarrow L^B$ *are antitonic and, furthermore, for all* $f \in L^B$ *and* $g \in L^A$ *we have* $f \subseteq \downarrow\uparrow f$ *and* $g \subseteq \uparrow\downarrow g$*.*

Definition 4. *Consider an* L*-fuzzy context* $C = \langle B, A, r\rangle$*. An* L*-fuzzy set of objects* $f \in L^B$ *(resp. an* L*-fuzzy set of attributes* $g \in L^A$*) is said to be* ***closed in*** $\mathbf{C}$ *iff* $f = \downarrow\uparrow f$ *(resp.* $g = \uparrow\downarrow g$*).*

Lemma 2. *Under the conditions of Lemma 1, the following equalities hold for arbitrary $f \in L^B$ and $g \in L^A$, $\uparrow f = \uparrow\downarrow\uparrow f$ and $\downarrow g = \downarrow\uparrow\downarrow g$, that is, both $\downarrow\uparrow f$ and $\uparrow\downarrow g$ are closed in C.*

Definition 5. *An L-**fuzzy concept** is a pair $\langle f, g\rangle$ such that $\uparrow f = g, \downarrow g = f$. The first component f is said to be the **extent** of the concept, whereas the second component g is the **intent** of the concept.*

The set of all L-fuzzy concepts associated to a fuzzy context (B, A, r) will be denoted as L-$FCL(B, A, r)$.

An ordering between L-fuzzy concepts is defined as follows: $\langle f_1, g_1\rangle \le \langle f_2, g_2\rangle$ if and only if $f_1 \subseteq f_2$ if and only if $g_1 \supseteq g_2$.

Proposition 1. *The poset $(L\text{-}FCL(B, A, r), \le)$ is a complete lattice where*

$$\bigwedge_{j\in J} \langle f_j, g_j\rangle = \Big\langle \bigwedge_{j\in J} f_j, \uparrow \big(\bigwedge_{j\in J} f_j\big)\Big\rangle \quad \textit{and} \quad \bigvee_{j\in J} \langle f_j, g_j\rangle = \Big\langle \downarrow \big(\bigwedge_{j\in J} g_j\big), \bigwedge_{j\in J} g_j\Big\rangle$$

3 Other operations on an *L*-context

The corresponding notions of negation, disjunction and complement on an L-context are introduced below.

Definition 6. *Let us consider a unary operator* negation *and a binary* disjunction *operator on the underlying structure of truth values L as follows:*

1. *Negation $\neg\colon L \to L$ is defined by $\neg(l) = \neg l = l \to 0$*
2. *Disjunction $\ltimes\colon L \times L \to L$ is defined by $l_1 \ltimes l_2 = \neg l_1 \to l_2$*

Some of the properties of negation appear in the following lemma.

Lemma 3 (Bělohlávek [5]). *For any $a, b, c \in L$ the following holds.*

1. $a \le \neg b \iff a \otimes b = 0$
2. $a \otimes \neg a = 0$
3. $a \le \neg\neg a$
4. $\neg 0 = 1$
5. $\neg a = \neg\neg\neg a$
6. $a \to b \le \neg b \to \neg a$
7. $a \le b \Longrightarrow \neg b \le \neg a$
8. $\neg(a \vee b) = \neg a \wedge \neg b$

From Property 6 above and the definition of disjunction, we can see that disjunction needs not be, in general, commutative. However, this property will be very important for the definition and properties of direct product of two L-contexts. Notice that commutativity will hold if the *law of double negation* ($\neg\neg a = a$) holds. The following result states some properties of residuated lattices satisfying double negation.

Proposition 2 (Bělohlávek [5]). *If a residuated lattice satisfies the law of double negation then it also satisfies the following conditions:*

1. $l \to k = \neg(k \otimes \neg l)$
2. $\neg(\bigwedge_{i\in I} l_i) = \bigvee_{i\in I} \neg l_i$
3. $l \to k = \neg k \to \neg l$

It is convenient here to recall that adding conditions of our underlying residuated lattice may change the class of structures we are working with. In particular, a residuated lattice satisfying the double negation law and divisibility (that is, $x \le y$ implies the existence of z such that $x = y \otimes z$), we are working with an MV-algebra. If divisibility is replaced by the fact that the product $\otimes$ coincides with the infimum of the lattice, then we are have just a Boolean algebra.

We finish this section with a specific notion of *complement* of a given L-fuzzy formal context.

Definition 7. *The complement of an L-fuzzy formal context is a formal context with the binary relation $\neg r$ defined by $\neg r(o,a) = r(o,a) \to 0$ for all $o \in B$ and $a \in A$. The uparrow and downarrow mappings on the complement are denoted by $\uparrow_\neg$ and $\downarrow_\neg$.*

Lemma 4. *Let $C = \langle B, A, r\rangle$ be an L-fuzzy formal context. For all objects $o, b \in B$ the inequality $\downarrow\uparrow(\chi_o)(b) \le \downarrow_\neg\uparrow_\neg(\chi_b)(o)$ holds. If, moreover, the law of double negation holds we have the equality $\downarrow\uparrow(\chi_o)(b) = \downarrow_\neg\uparrow_\neg(\chi_b)(o)$.*

Proof. In the following, we use the common notation χ_e to denote the characteristic function of an element e in a set:

$$\begin{aligned}
\downarrow\uparrow(\chi_o)(b) &= \bigwedge_{a\in A}(\uparrow(\chi_o)(a) \to r(b,a)) \\
&= \bigwedge_{a\in A}(\bigwedge_{c\in B}(\chi_o(c) \to r(c,a)) \to r(b,a)) \\
&= \bigwedge_{a\in A}((\bigwedge_{c\in B, c\neq o}(\chi_o(c) \to r(c,a)) \wedge (\chi_o(o) \to r(o,a))) \to r(b,a)) \\
&= \bigwedge_{a\in A}((\bigwedge_{c\in B, c\neq o}(0 \to r(c,a)) \wedge (1 \to r(o,a))) \to r(b,a)) \\
&= \bigwedge_{a\in A}((1 \wedge (1 \to r(o,a))) \to r(b,a)) \\
&= \bigwedge_{a\in A}((1 \to r(o,a)) \to r(b,a)) \\
&= \bigwedge_{a\in A}(r(o,a) \to r(b,a)) \\
&\overset{*}{=} \bigwedge_{a\in A}(\neg r(b,a) \to \neg r(o,a)) = \cdots = \downarrow_\neg\uparrow_\neg(\chi_b)(o)
\end{aligned}$$

Equality $(*)$ follows from the law of double negation, otherwise we can only obtain the inequality $\downarrow\uparrow(\chi_o)(b) \le \downarrow_\neg\uparrow_\neg(\chi_b)(o)$. □

4 *L*-Multifunctions and *L*-fuzzy relations

The definition of L-bonds is based on a suitable extension of the theory of multifunctions (also called, many-valued functions, or correspondences) whose notation and terminology is introduced below.

Definition 8. *An L-**multifunction** from X to Y is a mapping $\varphi\colon X \to L^Y$.*

*The **transposed** of an L-multifunction $\varphi\colon X \to L^Y$ is an L-multifunction ${}^{t}\varphi\colon Y \to {}^{X}L$ defined by ${}^{t}\varphi(y)(x) = \varphi(x)(y)$.*

The L-multifunction $\varphi\colon X \to L^Y$ can be extended to a mapping $\varphi^* : L^X \to L^Y$ by $\varphi^*(f)(y) = \bigvee_{x\in X}(f(x) \otimes \varphi(x)(y))$, for $f \in L^X$.

The set L-$Mfn(X, Y)$ of all the L-multifunctions from X to Y can be endowed with a poset structure by defining the ordering $\varphi_1 \leq \varphi_2$ as $\varphi_1(x)(y) \leq \varphi_2(x)(y)$ for all $x \in X$ and $y \in Y$.

The usual definition of curry and uncurry operations can be adapted to the framework of L-multifunctions as follows:

Definition 9. *Let us define for arbitrary L-multifunction $\varphi \in L$-$Mfn(X, Y)$ an L-fuzzy relation $\varphi^{\mathrm{r}} \in L^{X\times Y}$ defined by $\varphi^{\mathrm{r}}(x, y) = \varphi(x)(y)$ for all $(x, y) \in X \times Y$. For arbitrary L-fuzzy relation $r \in L^{X\times Y}$ lets define an L-multifunction from $r^{\mathrm{mfn}} : X \to L^Y$ defined by $r^{\mathrm{mfn}}(x)(y) = r(x, y)$.*

Finally, the notion of L-bond is given in the following definition:

Definition 10. *An L-**bond** between two formal contexts $C_1 = \langle B_1, A_1, r_1\rangle$ and $C_2 = \langle B_2, A_2, r_2\rangle$ is a multifunction $b\colon B_1 \to L^{A_2}$ satisfying the condition that for all $o_1 \in B_1$ and $a_2 \in A_2$ both $b(o_1)$ and ${}^{t}b(a_2)$ are closed L-fuzzy sets of, respectively, attributes in C_2 and objects in C_1. The set of all bonds from C_1 to C_2 is denoted as L-$Bonds(C_1, C_2)$.*

Lemma 5. *Let $\langle B_i, A_i, r_i\rangle$ be two L-fuzzy formal contexts for $i \in \{1, 2\}$, where L satisfies the double negation law. For all L-bonds $\beta \in L$-$Bonds(C_1, C_2)$ and for all objects $o_1 \in B_1$ the equation $\beta(o_1) = \beta^*(\downarrow_{\neg_1}\uparrow_{\neg_1}(\chi_{o_1}))$ holds.*

Proof. We will prove the two inequalities separately.

$$\begin{aligned}\beta(o_1)(a_2) &= \bigvee_{b_1\in B_1}(\beta(b_1)(a_2) \otimes \chi_{o_1}(b_1))\\ &\leq \bigvee_{b_1\in B_1}(\beta(b_1)(a_2)\otimes \downarrow_{\neg_1}\uparrow_{\neg_1}(\chi_{o_1})(b_1)) = \beta^*(\downarrow_{\neg_1}\uparrow_{\neg_1}(\chi_{o_1}))(a_2)\end{aligned}$$

For the other inequality, consider the following chain

$$\begin{aligned}
\beta^*(\downarrow_{\neg_1}\uparrow_{\neg_1}(\chi_{o_1}))(a_2) &= \bigvee_{b_1\in B_1}(\beta(b_1)(a_2)\otimes\downarrow_{\neg_1}\uparrow_{\neg_1}(\chi_{o_1})(b_1))\\
&\overset{*}{=} \bigvee_{b_1\in B_1}(\beta(b_1)(a_2)\otimes\downarrow_1\uparrow_1(\chi_{b_1})(o_1))\\
&= \bigvee_{b_1\in B_1}({}^{\mathrm{t}}\beta(a_2)(b_1)\otimes\bigwedge_{a_1\in A_1}(\uparrow_1(\chi_{b_1})(a_1)\to r_1(o_1,a_1)))
\end{aligned}$$

${}^{\mathrm{t}}\beta(a_2)$ is a closed L-set in B_1, then ${}^{\mathrm{t}}\beta(a_2)(b_1)=\downarrow_1(g)(b_1)$ for some $g\in L^{A_1}$

$$\begin{aligned}
&= \bigvee_{b_1\in B_1}(\downarrow_1(g)(b_1)\otimes\bigwedge_{a_1\in A_1}((1\to r_1(b_1,a_1))\to r_1(o_1,a_1)))\\
&= \bigvee_{b_1\in B_1}(\bigwedge_{a_1\in A_1}g(a_1)\to r_1(b_1,a_1))\otimes\bigwedge_{a_1\in A_1}(r_1(b_1,a_1)\to r_1(o_1,a_1)))\\
&\overset{\star}{=} \bigvee_{b_1\in B_1}\bigwedge_{a_1\in A_1}((g(a_1)\to r_1(b_1,a_1))\otimes(r_1(b_1,a_1)\to r_1(o_1,a_1)))\\
&\leq \bigvee_{b_1\in B_1}\bigwedge_{a_1\in A_1}(g(a_1)\to r_1(o_1,a_1)) =\\
&= \bigvee_{b_1\in B_1}\downarrow_1(g)(o_1) = \bigvee_{b_1\in B_1}{}^{\mathrm{t}}\beta(a_2)(o_1) = \bigvee_{b_1\in B_1}\beta(o_1)(a_2)\\
&= \beta(o_1)(a_2)
\end{aligned}$$

where $(\star)$ follows from the inequality $(k\to l)\otimes(l\to m)\leq k\to l$ which holds for all $k,l,m\in L$. □

5 Direct product of two L-fuzzy contexts

Here we introduce the corresponding extension of the notion of direct product of two L-fuzzy contexts.

Definition 11. *The* direct product *of two L-fuzzy contexts $C_1=\langle B_1,A_1,r_1\rangle$ and $C_2=\langle B_2,A_2,r_2\rangle$ is an L-fuzzy context $C_1\Delta C_2=\langle B_1\times A_2, A_1\times B_2,\Delta\rangle$, such that $\Delta((o_1,a_2),(a_1,o_2))=\neg r_1(o_1,a_1)\to r_2(o_2,a_2)$.*

The following result states properties of the just defined direct product of L-fuzzy contexts.

Lemma 6. *Let $C_1=\langle B_1,A_1,r_1\rangle$ and $C_2=\langle B_2,A_2,r_2\rangle$ be two L-fuzzy contexts, where L satisfies the double negation law. Given two arbitrary L-multifunctions $\varphi\colon B_1\to L^{A_2}$ and $\psi\colon A_2\to L^{B_1}$, for all $o_1,o_2\in B_1$ and $a_1,a_2\in A_2$ the following equalities hold*

$$\uparrow_\Delta(\varphi^{\mathrm{r}})(o_2,a_1)=\downarrow_2(\varphi^*(\downarrow_{\neg_1}(\chi_{a_1})))(o_2)=\uparrow_1({}^{\mathrm{t}}\varphi^*(\uparrow_{\neg_2}(\chi_{o_2})))(a_1)$$
$$\downarrow_\Delta(\psi^{\mathrm{r}})(o_1,a_2)=\uparrow_2(\psi^*(\uparrow_{\neg_1}(\chi_{o_1})))(a_2)=\downarrow_1({}^{\mathrm{t}}\psi^*(\downarrow_{\neg_2}(\chi_{a_2})))(o_1)$$

Proof. Consider the following chain of equalities:

$$
\begin{aligned}
&\uparrow_{\Delta} (\varphi^{\mathrm{r}})(o_2, a_1)\\
&= \bigwedge_{(o_1,a_2)\in B_1\times A_2} (\varphi^{\mathrm{r}}(o_1, a_2) \to \Delta((o_1, a_2), (o_2, a_1)))\\
&= \bigwedge_{(o_1,a_2)\in B_1\times A_2} (\varphi^{\mathrm{r}}(o_1, a_2) \to (\neg r_1(o_1, a_1) \to r_2(o_2, a_2)))\\
&= \bigwedge_{(o_1,a_2)\in B_1\times A_2} ((\varphi^{\mathrm{r}}(o_1, a_2) \otimes \neg r_1(o_1, a_1)) \to r_2(o_2, a_2))\\
&= \bigwedge_{(o_1,a_2)\in B_1\times A_2} ((\varphi^{\mathrm{r}}(o_1, a_2) \otimes (1 \to \neg r_1(o_1, a_1))) \to r_2(o_2, a_2))\\
&= \bigwedge_{(o_1,a_2)\in B_1\times A_2} ((\varphi^{\mathrm{r}}(o_1, a_2) \otimes \bigwedge_{t_1\in A_1} (\chi_{a_1}(t_1) \to \neg r_1(o_1, t_1))) \to r_2(o_2, a_2))\\
&= \bigwedge_{(o_1,a_2)\in B_1\times A_2} ((\varphi^{\mathrm{r}}(o_1, a_2) \otimes \downarrow_{\neg_1} (\chi_{a_1})(o_1)) \to r_2(o_2, a_2))\\
&= \bigwedge_{o_1\in B_1} \bigwedge_{a_2\in A_2} ((\varphi^{\mathrm{r}}(o_1, a_2) \otimes \downarrow_{\neg_1} (\chi_{a_1})(o_1)) \to r_2(o_2, a_2))\\
&= \bigwedge_{o_1\in B_1} \bigwedge_{a_2\in A_2} (\downarrow_{\neg_1} (\chi_{a_1})(o_1) \otimes (\varphi(o_1)(a_2)) \to r_2(o_2, a_2))\\
&= \bigwedge_{o_1\in B_1} \downarrow_2 (\downarrow_{\neg_1} (\chi_{a_1})(o_1) \otimes \varphi(o_1))(o_2)\\
&= \downarrow_2 (\bigvee_{o_1\in B_1} (\downarrow_{\neg_1} (\chi_{a_1})(o_1) \otimes \varphi(o_1)))(o_2)\\
&= \downarrow_2 (\varphi(\downarrow_{\neg_1} (\chi_{a_1})))(o_2)
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
\uparrow_{\Delta} (\varphi^{\mathrm{r}})(o_2, a_1) &= \bigwedge_{(o_1,a_2)\in B_1\times A_2} (\varphi^{\mathrm{r}}(o_1, a_2) \to (\neg r_1(o_1, a_1) \to r_2(o_2, a_2)))\\
&= \bigwedge_{(o_1,a_2)\in B_1\times A_2} ({}^{\mathrm{t}}\varphi(a_2)(o_1) \to (\neg r_2(o_2, a_2) \to r_1(o_1, a_1)))\\
&\;\;\vdots\\
&= \uparrow_1 ({}^{\mathrm{t}}\varphi^{*}(\uparrow_{\neg_2} (\chi_{o_2})))(a_1)
\end{aligned}
$$

□

6 *L*-bonds vs direct products of *L*-fuzzy contexts

The main contribution of the paper is presented in this section, in which a relationship between *L*-bonds and extents of direct products of *L*-fuzzy contexts is drawn by the following theorem.

Theorem 1. *Let $C_i = \langle B_i, A_i, r_i \rangle$ be L-fuzzy contexts for $i \in \{1,2\}$, where L satisfies the double negation law. Let $\beta \in L\text{-}Mfn(B_1, A_2)$. Then:*

1. *If β^{r} is an extent of $C_1 \Delta C_2$, then $\beta \in L\text{-}Bond(C_1, C_2)$.*
2. *If $\beta \in L\text{-}Bond(C_1, C_2)$ and*

$$\beta^*(\downarrow_{\neg_1}\uparrow_{\neg_1}(\chi_{o_1}))(a_2) = \bigwedge_{a_1 \in A_1} (\uparrow_{\neg_1}(\chi_{o_1})(a_1) \to \uparrow_2\downarrow_2 (\beta^*(\downarrow_{\neg_1}(\chi_{a_1})))(a_2))$$

then ${}^{\mathrm{r}}\beta$ is an extent of $C_1 \Delta C_2$.

Proof. 1. For the first item, let β be an extent of $C_1 \Delta C_2$, then we know that $\beta(o_1)(a_2) = \downarrow_\Delta\uparrow_\Delta (\beta^{\mathrm{r}})(o_1, a_2)$

Let us write $\uparrow_\Delta (\beta^{\mathrm{r}})^{\mathrm{mfn}} = \psi$, then

$$\beta(o_1)(a_2) = \downarrow_\Delta (\psi)(o_1, a_2) = \uparrow_2 (\psi^*(\uparrow_{\neg_1}(\chi_{o_1})))(a_2)$$

As a result, $\beta(o_1)$ is a closed L-set from L^{A_2}.

Similarly, we have that ${}^{\mathrm{t}}\beta(a_2)(o_1) = \downarrow_1 ({}^{\mathrm{t}}\psi^*(\downarrow_{\neg_2}(\chi_{a_2})))(o_1)$. Hence ${}^{\mathrm{t}}\beta(a_2)$ is a closed L-set of objects from L^{B_1}.

2. The proof for the second item is as follows:

$$\begin{aligned}
&\bigwedge_{a_1 \in A_1} (\uparrow_{\neg_1}(\chi_{o_1})(a_1) \to \uparrow_2\downarrow_2 (\beta^*(\downarrow_(\chi_{a_1})))(a_2)) = \\
&\quad = \bigwedge_{a_1 \in A_1} (\uparrow_{\neg_1}(\chi_{o_1})(a_1) \to \bigwedge_{o_2 \in B_2} (\downarrow_2 (\beta^*(\downarrow_{\neg_1}(\chi_{a_1})))(o_2) \to r_2(o_2, a_2))) \\
&\quad = \bigwedge_{a_1 \in A_1} \bigwedge_{o_2 \in B_2} (\uparrow_{\neg_1}(\chi_{o_1})(a_1) \to (\downarrow_2 (\beta^*(\downarrow_{\neg_1}(\chi_{a_1})))(o_2) \to r_2(o_2, a_2))) \\
&\quad = \bigwedge_{a_1 \in A_1} \bigwedge_{o_2 \in B_2} ((\uparrow_{\neg_1}(\chi_{o_1})(a_1) \otimes \downarrow_2 (\beta^*(\downarrow_{\neg_1}(\chi_{a_1})))(o_2)) \to r_2(o_2, a_2)) \\
&\quad = \bigwedge_{o_2 \in B_2} (\bigvee_{a_1 \in A_1} (\uparrow_{\neg_1}(\chi_{o_1})(a_1) \otimes \downarrow_2 (\beta^*(\downarrow_{\neg_1}(\chi_{a_1})))(o_2)) \to r_2(o_2, a_2)) \\
&\quad = \uparrow_2 (\bigvee_{a_1 \in A_1} (\uparrow_{\neg_1}(\chi_{o_1})(a_1) \otimes \downarrow_2 (\beta^*(\downarrow_{\neg_1}(\chi_{a_1}))))(a_2) \\
&\quad = \uparrow_2 (\bigvee_{a_1 \in A_1} (\uparrow_{\neg_1}(\chi_{o_1})(a_1) \otimes (\uparrow_\Delta (\beta^{\mathrm{r}}))^{\mathrm{mfn}}(a_1))(a_2) \\
&\quad = \uparrow_2 ((\uparrow_\Delta (\beta^{\mathrm{r}}))^{\mathrm{mfn}*}(\uparrow_{\neg_1}(\chi_{o_1})))(a_2) \\
&\quad = \downarrow_\Delta\uparrow_\Delta (\beta^{\mathrm{r}})(o_1, a_2) \\
&\quad \overset{\star}{=} \beta^*(\downarrow_{\neg_1}\uparrow_{\neg_1}(\chi_{o_1}))(a_2) = \beta(o_1)(a_2)
\end{aligned}$$

where ($\star$) follows, firstly, from the hypothesis, which states that it equals to $\beta^*(\downarrow_{\neg_1}\uparrow_{\neg_1}(\chi_{o_1}))(a_2)$ and, as $\beta \in L\text{-}Bond(C_1, C_2)$, by Lemma 5. □

7 Conclusions and future work

We have introduced an adequate generalization of the study of L-bonds as morphisms among contexts, initiated in [14], by showing how the classical relationships between bonds and contexts can be lifted to a more general framework.

The contribution seems to pave the way towards determining possible categories on which to model knowledge transfer and information sharing. Other steps have been given in [12, 13] where the category of L-Chu correspondences has been considered. However, much work still has to be done.

A thorough study of the properties of the extended categorical framework of Chu correspondences and L-Chu correspondences is needed, in order to identify their natural interpretation within the theory of knowledge representation.

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