The Mathematical in Music Thinking

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Abstract. *"The Mathematical in music thinking"* is based on Heidegger's understanding of "the Mathematical" as the basic assumption of the knowledge of the things. Heidegger's ideas are combined with Peirce's classification of sciences, in particular, to distinguish between the Mathematical from the less abstract logical thinking and the more abstract mathematical thinking. The aim of this paper is to make understandable the role of the Mathematical in music. The paper concentrates on three domains: the rhythmic of music, the doctrine of music forms, and the theory of tonal systems. The theoretical argumentations are assisted by musical examples: the Adagio of Mozart's string quartet C major (KV 465), the second movement of Webern's Symphony op.21, and a cadence illustrating the problem of the harmony of second degree.

1 Music Thinking and The Mathematical

"Musica est exercitium arithmeticae occultum animi" (*"Music is a hidden arithmetical exercise of the soul"*) - this statement was written by the philosopher, mathematician, and scientist Gottfried Wilhelm Leibniz on April 17, 1712, in a letter to the mathematician and diplomat Christian von Goldbach. Leibniz referred with his statement to the astonishing *phenomenon of the correspondence between musical tones and numbers* which has been already demonstrated by the pythagoreans on their monochord. This phenomenon has been extensively described by the German musicologist Martin Vogel in his book *"Die Lehre von den Tonbeziehungen"*; there he writes: "Each interval used in music corresponds to a certain numerical proportion and, since each melody and each harmonic connection can be composed by numerically described intervals, each composition can finally be understood and analytically recognized as an arrangement of uniquely determined relations of numbers" ([18], p.9).

If one wants to comprehensively understand the role of mathematics in music thinking, then the numerical relations in music compositions pointed out by Vogel do not suffice. In particular, the numerical relations cannot suitably grasp the more extended set semantics basic for modern mathematics. For our theme we use the understanding of *"the Mathematical"* which Martin Heidegger worked out in his 1935/36 lecture on "Basic Questions of Metaphysics" (published in [11]). For Heidegger "the Mathematical" is not derivable out of mathematics, but mathematics itself is at the time a historically, socially, and culturally determined formation abstracted from the Mathematical. Heidegger deduced his

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understanding of "the Mathematical" from the ancient Greeks: $\tau\grave{\alpha}\ \mu\alpha\vartheta\acute{\eta}\mu\alpha\tau\alpha$ means "the learnable". Learning the learnable is a kind of "taking", by which the taker takes only such things which, strictly speaking, he already has. According to Heidegger it follows: "$\tau\grave{\alpha}\ \mu\alpha\vartheta\acute{\eta}\mu\alpha\tau\alpha$, the *Mathematical*, is what of the things we actually already know, which we therefore do not first take out of the things, but which we already bring with us in a certain way" ([11], p.57); or phrased in another way: "The Mathematical is that basic position to the things by which we take on the things according to that which the things have already been given to us. The Mathematical is therefore the basic assumption of the knowledge of the things" ([11], p.58). For Heidegger this makes clear the central significance of the Mathematical for modern thinking, because *"a will of reformation and self-foundation of the knowledge form as such"* lies in the character of the Mathematical as distinctive conception ([11], p.75).

But how can we recognize the Mathematical? A promising approach is to abstract logical forms of thinking to mathematical forms of thinking which gives rise to rich mathematical theory developments retroacting, in particular, the logical forms and in this way enriching also the logical thinking (cf. [23]). To capture the Mathematical in music thinking, it suggests itself to identify first of all *the logical in music*, for instance in a manner as articulated by the musicologist Hans-Peter Reineke in referring to musical hearing; he writes: "Certain regulatives in musical hearing constitute and preserve *music as a logical being* that must sound plausibly out of itself if it shall be accepted" [17]. During the ending 18th century the term "musical logic" was linked to the idea "that music is an art which is autonomous, resting in itself, and submitted only to its own law of form; in particular, its right to exist needs not to be justified extramusically" ([3], p.66). But, inspite of numerous efforts (here, first of all, the musicologist Hugo Riemann has to be named), a musical logic has never been really established in musicology. Nevertheless, to identify the Mathematical in music thinking, the connection between logical and mathematical thinking shall be discussed more extensively.

The philosopher and scientist Charles Sanders Peirce has convincingly described the connection between logical and mathematical thinking in the frame of his philosophy of science. In his classification of sciences from 1903 ([16], 258ff.), in which he ordered the sciences by the degree of their abstractness, mathematics as the most abstract science of all sciences is positioned at the most abstract level. As the only hypothetical science, mathematics has the task to develop a cosmos of forms of potential realities. All other sciences, under which philosophy is the most abstract, relate to actual realities. According to Peirce's classification, philosophy partitions into phenomenology, normative science, and metaphysics while normative science divides further into esthetics, ethics, and logic. Musicology has to be classified - such as history - under the descriptive science. In Peirce's classification the sciences are ordered in a manner that each science

– refers, according to its general principles, exclusively to the sciences which are more abstract than itself, and

– makes use of examples and specific facts elaborated by sciences which are less abstract than then the considered science.

For instance, logic as the third part of normative science is supposed to refer to ethics, esthetics, phenomenology, and mathematics concerning its general principles, and gains its actually real contents from metaphysics and the special sciences, particularly also from musicology. On the other hand, musicology can benefit from the manifoldness of the forms of logical and mathematical thinking.

As already pointed out, Heidegger does not view *"the Mathematical"* as part of mathematics, but views mathematics as an abstraction of the Mathematical, respectively. Thus, it seems very likely to locate the Mathematical within the phenomenology which is the initial part of philosophy in Peirce's classification of sciences ([16], p.258ff.). According to Peirce, the general task of phenomenology is to investigate the universal qualities of the phenomenons in their immediate character. Heidegger's conceptions of thingness can be understood as such universal qualities of phenomenons. This becomes more clear by the following determination of the nature of the Mathematical which has been summarized by Heidegger in his book [11] on p.71f:

1. The Mathematical is a conception of *thingness* leaping virtually over its things.
2. This conception determines what the *things* are considered for, as what they and how they should be acknowledged in advance.
3. The conception of the Mathematical is an *axiomatic anticipation* in the nature of the things tracing out how each thing and each relationship between those things are formed.
4. This formation offers the *scale* for delimiting the domain which embraces in future all things of such nature.
5. The axiomatically determined domain now demands for the things belonging to it an *accessibility* suitable alone for the axiomatically predetermined things.

For getting a better understanding of Heidegger's conception of the Mathematical, it might be helpful to discuss Heidegger's summary with respect to an example. Let us choose the space in which we live. Our understanding of the space is quite supported by our experiences with the bodies in the space so that we can rephrase Heidegger's five statements concerning the space of bodies as follows:

1. The conception of *space* leaping over its bodies is a model of the Mathematical (which has been abstracted mathematically to the real vector space).
2. This conception determines what the *bodies* are considered for, as what they and how they should be acknowledged in advance (which can be supported by representing the bodies mathematically using bounded connected subsets of the real vector space).
3. The conception of the Mathematical is an *axiomatic anticipation* in the nature of the bodies tracing out how each body and each relationship between

those bodies are formed (which become mathematically descriptive by algebraic terms).
4. This formation offers the *scale* for delimiting the domain which embraces in future all bodies of such nature (in particular, this allows to measure bodies mathematically).
5. The axiomatically determined spacial domain now demands for the bodies belonging to it an *accessibility* suitable alone for the axiomatically predetermined bodies (which can be mathematically abstracted within the axiomatically defined real vector space).

Let us record for this paper that the Mathematical as part of phenomenology is less abstract than mathematics, but is more abstract than logic, the third subpart of normative science. For investigating the Mathematical in music thinking, it is important to understand the relationships between Mathematical and logical thinking. Peirce convincingly explains the close connection between logical and mathematical thinking in his Cambridge Conferences Lectures from 1898, which have only completely be published, with 100 pages introduction and commentary, in 1992 under the title "Reasoning and the Logics of Things" [15]. Without pointing in details to Peirce's explanations, it shall be attempted in the following to demonstrate an analogous connection between the forms of music thinking and the forms of the Mathematical with its abstractions in mathematics thinking. The manifoldness of music thinking, in which we would have to investigate the Mathematical, cannot exhaustively be discussed in this contribution. Therefore we shall concentrate on forms of thinking about the rhythmic of music, the doctrine of music forms, and the theory of tone systems.

2 The Mathematical in the Rhythmic of Music

By Riemann's Music Encyclopaedia, *rhythm* has to be understood as an autonomous principle of form and order which is characterized on the one hand by regularity and relationship to a fixed tempo, on the other hand by grouping, subdivision, and alternation. In this conceptual characterization, first of all

- the *"uniformity of parts"* ,
- the *"succession of parts"*, and
- the *"distinctness of parts"*

have entered in music thinking as basic forms of thinking of the Mathematical. In the case of rhythmic, these forms of thought become forms of mathematics if uniformity, succession, and distinctness of rhythm-parts are defined in the sense of an *established semantics of mathematics*. The metric fixation of rhythmic in musical notation may definitely be understood as such a semantically abstracting mathematization. However the musical interpretations usually liberate from the rigid mathematical structure by their agogics and accentuations. Therefore the Mathematical does not disappear by mathematizing the rhythmic, but keeps preserved in its autonomous independence.

Fig. 1. The Adagio of Mozart's string quartet C major (KV 465)

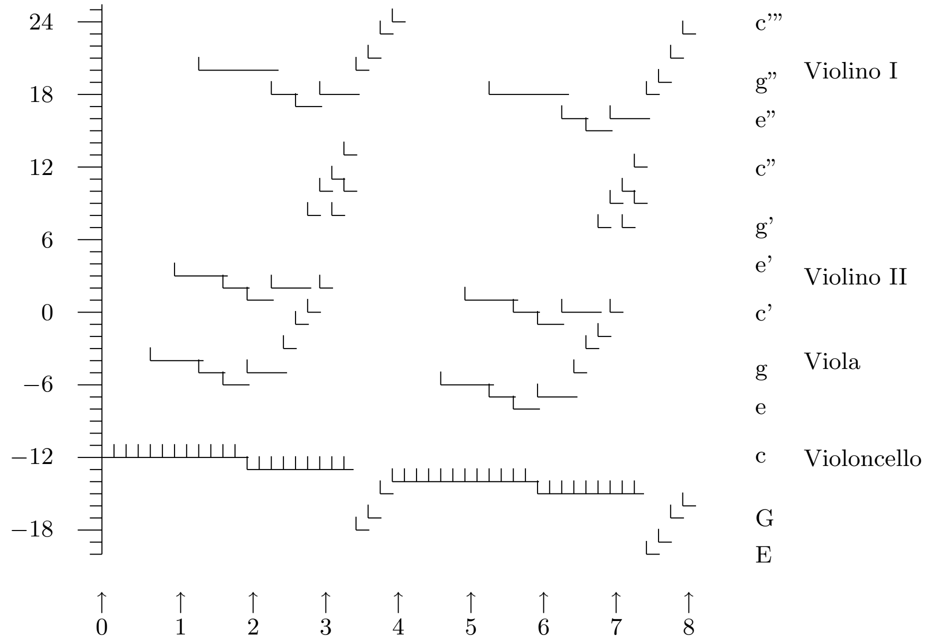

Fig. 2. A mathematical representation of the first eight bars of the Adagio of Mozart's string quartet C major (KV 465)

The interplay between the Mathematical in music and the mathematization of music shall be demonstrated here by the Adagio of Mozart's string quartet C major (KV 465). The score of the Adagio - presented in Fig. 1 - shows that the Adagio consists only of 22 bars in which astonishingly many dissonances occur, but which finally leads to the light C major clearness of the following Allegro.

The result of a mathematization of the first eight bars of the Adagio is shown in Fig. 2. The presented mathematical structure shall be considered as embedded into a two-dimensional real vector space. The part of its vertical axis from -20 to 25 is visible on the left of the diagram (the numbers -18, -12, -6, 0, 6, 12, 18, 24 shall help to identify the integer locations on the vertical axis). There is a one-to-one correspondence between the integers of the vertical axis from -20 to 25 and the tones of the chromatic scale from E to $c'''\#$. Fig. 2 indicates the part of this correspondence which horizontally links the numbers $-20 < -17 < -12 < -8 < -5 < 0 < 4 < 7 < 12 < 16 < 19 < 24$ to the tones of the C major triad $E < G < c < e < g < c' < e' < g' < c'' < e'' < g'' < c'''\#$

The location of the integers 0, 1, ... , 8 on the (imaginary) horizontal axis are indicated by the numbers on the bottom of the diagram (the smallest unit for the horizontal numbers is one sixth, in numerals: 1/6). The horizontal straight line segments on the right of the vertical axis, closed on the left end and open on the right end, represent the sounds of the four instruments with their pitches, respectively (the pitch of such a line segment is determined by the height of the line segment measured by the vertical axis). The small vertical line segments on the right of the vertical axis and the line segment between the points (0,-12) and (0,-11) indicate the beginning of the sound belonging to the horizontal line segment connected at the bottom of that small vertical line segment. The union of all those line segments can be divided into four disjoint subsets corresponding exactly to the four instruments Violino I, Violino II, Viola, and Violoncello (notice that the representations of the sounds beginning at the points (18/6,2), (19/6,8), (20/6,10) and (42/6,0), (43/6,6), (44/6,8) belong to the Viola subset, but not to the Violino II subset). Thus, the structure of those four subsets determines the mathematical representation of the first eight bars of the Adagio. It is not difficult to extend this representation to a mathematical representation of the whole Adagio.

Although the discussed mathematical description of the tones of the Adagio by their pitch, length, location, and instrument are in one-to-one correspondence to the notes of the score presented in Fig. 1, there are more signatures in the score concerning tempo, loudness, crescendo, and bows which are not mathematized. Above all the expressive interpretations of a score by rhythm, agogics, accentuations etc. are far away from a meaningful mathematization. That, in particular, the rhythm evades any mathematical description becomes clear by the following quotation: "The rhythm comprises the order, division, and meaningful arrangement of the time development of sound events. In spite of the tendency, created by the rhythm, to return to the same or the similar, the rhythm should not be confused with the metre and beat because just the vivid differences of the courses of time make possible the musical manifoldness of the rhythms which first of all appear through graded durations of sounds and accents, but also through melodic movements, changing sounds and tone colours, changes of tempo and loudness, phrasing and articulation" ([2], p.656).

3 The Mathematical in the Doctrine of Music Forms

According to the Composer György Ligeti: "The combination of association, abstraction, remembrance, and prevision let only actually achieve the suggestiveness which makes possible the conception of a musical form" ([1], p.9). Without the principle of order, the musical forms would be neither communicable nor apperceivable. Clear orders and relationships are a criterion of its conceivability and indispensible assumption for its understanding. The smallest units of musical sense are the so-called *"motives"* which are understood as the smallest meaningful elements of musical compositions. Motives join up with their own transformations and other motives to larger parts which might be again only parts of a larger whole (cf. [1], p.16f).

For understanding the Mathematical in music forms, it might be helpful to analyse the multitude of music forms in the Adagio presented in Fig. 1 (cf. ([14], p.446). As a whole the music form is an *introduction* to the Allegro, the first movement of the C major string quartet (KV 465). The introduction divides into two parts each of which has 11 bars; the *first part* is polyphonic, the *second* *part* is homophonic. The violoncello starts the Adagio with eighth notes repeated through all the eleven bars of the first part, interrupted only by a *four notes motive* chromatically ascending at the end of the fourth bar and the eighth bar, respectively. After the first four eigth notes of the violoncello the other three instruments present a *theme* which divides into two motives each of which consisting of four notes, where the viola starts at the end of the first bar, the violino II one quarter note later, and the violino I again one quarter note later. The first chord of the four instruments combining the notes c - g - e'♭ - a" contains the two surprising dissonances g - a" and e'♭ - a" and allowed in the following further dissonances until the second motive occurs in combing consonant chords. Starting from the fifth bar, the first four bars are repeated always a major note downwards. The last three bars of the first part of the Adagio function as a bridge to the second part in which the four instruments play the same role between each other in diminishing the motives.

The example shows that the *mathematization of music forms* can use in addition to the descriptive dimensions pitch, length, location, and instrument also the dimension *"music form"*. In our example Fig. 1 we can consider as music forms the whole Adagio, the disjunctive two parts of the Adagio which cover the Adagio, smaller meaningful parts such as periods, themes, phrases, motives, scales, harmonies, chords, tones etc. Many of those music forms of the Adagio can be mathematically represented by a subset of the two-dimensional vector space sketched in Fig. 2; for instance:

- the first motive of the theme presented first for the viola,
- the first theme presented first for the viola,
- the first motive of the theme presented first for the violino II,
- the first theme presented first for the violino II,
- the first motive of the theme presented first for the violino I,
- the first theme presented first for the violino I,
- the first motive of the theme presented secondly for the viola,
- the first theme presented first for the viola,
- the first motive of the theme presented secondly for the violino II,
- the first theme presented first for the violino II,
- the first motive of the theme presented secondly for the violino I,
- the first theme presented first for the violino I,
- the first four tone motive ending with B presented for the violoncello,
- the second four tone motive ending with B presented for the violoncello.

The mathematical description of music forms may extend the mathematization of structures determined by the dimensions of pitch, length, location, and instrument as, for example, presented in Fig. 2. Nevertheless, the expressive interpretations of musical scores are still not in reach to be completely mathematized. Thus, there is still quite a distance between the Mathematical and the more abstract mathematization, but further attempts of diminishing the distance can be elaborated of which two approaches shall be briefly mentioned.

In the doctrine of music forms, symmetries play a special role for which the form of thinking *"equality of parts as expression of a whole"* (cf. [19]) can be assumed to belong to the Mathematical. This phenomenological form of thinking finds its abstraction in mathematics by the mathematical concepts of *symmetry transformation*" and *"symmetry group*", respectively. A direct correspondence between the phenomenological and the mathematical form of thinking regarding compositions is almost only given by strong canons. But if one weakens the mathematical concept of symmetry transformation to a concept of partial symmetry transformation, then considerably more correspondencies could be identified.

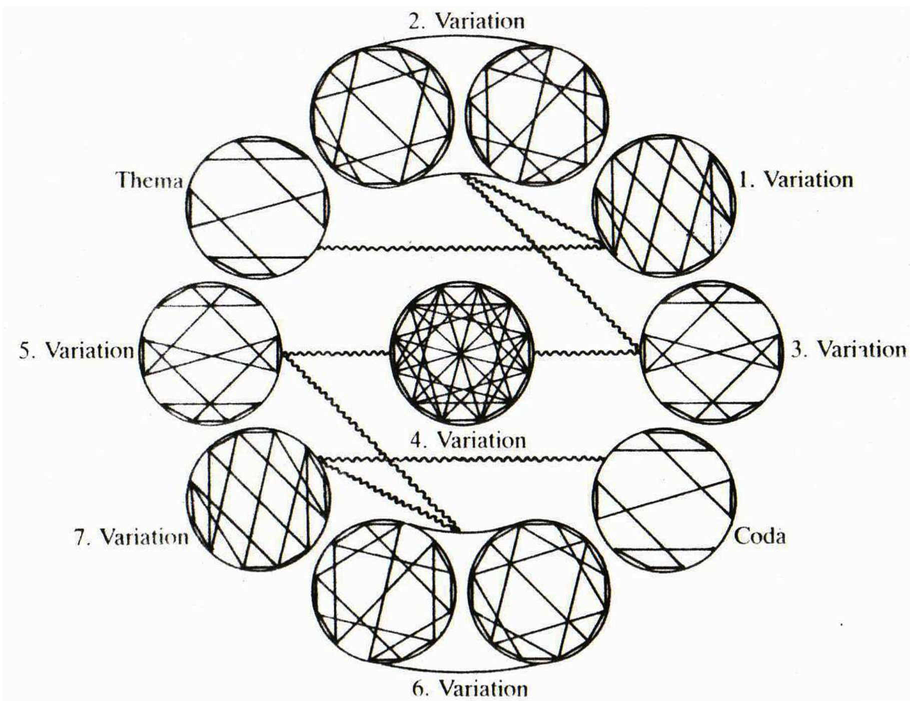

Fig. 3. The symmetry structure of the second movement of Anton Webern's Symphony op. 21

As another generalization of the mathematical form of symmetry, the twelve-tone music used more general symmetries which view octave tones to be structurally identified. The example shown in Fig. 3 represents twelve tone rows by a sequence of eleven straight sections on a circle. Each circle presents at least one symmetry and all circles together are arranged in such a way that a 180° rotation maps the total picture onto itself. Musically this indicates that the total symphony is a transposition of its retrogression.

The composer Fred Lerdahl and the linguist Ray Jackendoff have elaborated a much more far-reaching approach to formally grasping forms of music which was published in their book *"A generative theory of tonal music*" [13]. For this, they developed a generative grammar of music, which was inspired by Chomsky's linguistic transformation grammar, but developed purely within music thinking.

As fundamental components of the musical understanding of a composition they considered grouping structures of subunits of the composition. For these grouping structures the form of thinking *"division of a whole into subunits"* can be assumed to belong to the Mathematical and abstracted to a mathematical structure of a weighted ordered set. Lerdahl and Jackendoff impressively demonstrate their theory by many examples, as fore instance by the beginning of Mozart's Symphony G minor, KV 550.

4 The Mathematical in the Theory of Tonal Systems

Tonal systems, which serve as foundation of music thinking, rest thoroughly on different forms of thinking of the Mathematical:

– Behind the *tonal system of the equal-tempered keyboard*, there is the form of thinking of a musical scale consisting of 7 white keys with the steps whole-whole-half-whole-whole-whole-half which are completed by 5 black keys to a musical scale with 12 half steps.
– The *tonal systems of musical instruments with finger-board* suggest a form of thinking which relates to finger positions; for example, the player of a violin thinks especially which finger has to be placed on which string in which position .
– The *tonal system of the names of tones* obtains its form by the names of the 12 octave tones c - c# - d - e♭ - e - f - f# - g - a♭ - a - b♭ - b which are rising by half-tone steps; adding # or ♭ to a tone name yields the name of a tone which is a half-tone higher or lower, respectively.
– The *tonal system of the standard notation* is founded on the form of the 5+5-line system with additional ledger lines, in which the tones are represented by note-heads with and without accidentals on and between the lines; the tone distances describable in this way are multiples of half-tone steps.
– The *harmonic tone system* extends the form of the tone system of tone names by adding integer exponents to the tone names; a tone name t^z represents a tone which is z-many syntonic commas higher or lower than the tone t^0, respectively (syntonic comma := 4 fifth − 2 octaves − 1 major third; multiplicatively, the syntonic comma is the frequency ratio 81 : 80 obtained by computing $((3:2)^4 : (2:1)^2) : (5:4))$ where the frequency ratio 3 : 2 represents the fifth, the ratio 2 : 1 the octave and the ratio 5 : 4 the major third).

Here only the *harmonic tone system* shall be further discussed. In Fig. 4, this system is represented by a tone net in just intonation which is freely generated by the perfect fifth 3 : 2 and the perfect major third 5 : 4 (modulo the octave 2 : 1). Leonhard Euler was the first who published such a tone net which he named *speculum musicum* [6]. Following Euler's idea, realizations of the harmonic tone system on musical instruments have been approached again and again (for an overview about those attempts see [18]). In particular, the

e^{-2}	b^{-2}	$f\#^{-2}$	$c\#^{-2}$	$g\#^{-2}$	$d\#^{-2}$	$a\#^{-2}$	$e\#^{-2}$	$b\#^{-2}$
c^{-1}	g^{-1}	d^{-1}	a^{-1}	e^{-1}	b^{-1}	$f\#^{-1}$	$c\#^{-1}$	$g\#^{-1}$
$a\flat^{0}$	$e\flat^{0}$	$b\flat^{0}$	f^{0}	c^{0}	g^{0}	d^{0}	a^{0}	e^{0}
$f\flat^{+1}$	$c\flat^{+1}$	$g\flat^{+1}$	$d\flat^{+1}$	$a\flat^{+1}$	$e\flat^{+1}$	$b\flat^{+1}$	f^{+1}	c^{+1}
$d\flat\flat^{+2}$	$a\flat\flat^{+2}$	$e\flat\flat^{+2}$	$b\flat\flat^{+2}$	$f\flat^{+2}$	$c\flat^{+2}$	$g\flat^{+2}$	$d\flat^{+2}$	$a\flat^{+2}$

Fig. 4. The tone net of the harmonic tone system

instrument MUTABOR should be mentioned which even allows to realize arbitrary mutating pitches of tones in just intonation, but also in any other form of intonation (see [7], [22]).

Although performing music pieces in just intonation is an ideal for many music ensembles (for instance for a string quartet), there are problems of being consistent with the intonation. This shall be briefly explained by the so-called *Problem of the Harmony of Second Degree* illustrated in the harmonic tone system shown in Fig. 5 (cf. [21], p.197f). The figure represents a musical cadence formed by five perfect triads starting with the major triad $\underline{c}^0$ and ending with the major triad $\underline{c}^{-1}$. More precisely,

- the major triad $\underline{c}^0$ meets the major triad $\underline{f}^0$ in the note c,
- the major triad $\underline{f}^0$ meets the minor triad $\underline{d}^{-1}$ in the notes f^0 and a^{-1},
- the minor triad $\underline{d}^{-1}$ meets the major triad $\underline{g}^{-1}$ in the note d^{-1}, and
- the major triad $\underline{g}^{-1}$ meets the major triad $\underline{c}^{-1}$ in the note g^{-1}.

Playing a cadence as described above, musicians usually have the tendency to end with the same chord as they started with, i.e. with the major triad $\underline{c}^0$. Then, of course, they have to modify the pitches in between, but still to produce perfect triads. Cadences with such intonations defy convincing mathematization so that it would be interesting to find out how much *the Mathematical* could contribute to overcome those vaguenesses.

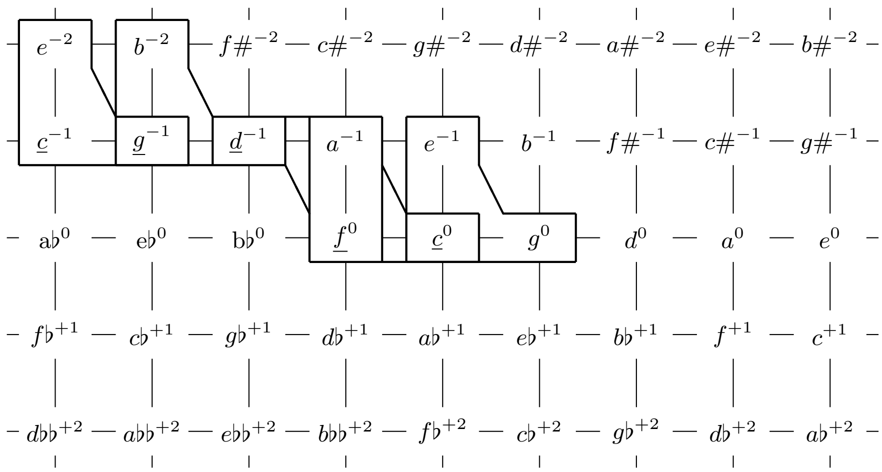

Fig. 5. A musical cadence leading from the major triad $\underline{c}^0$ in four steps via the major triad $\underline{f}^0$, minor triad $\underline{d}^{-1}$, and the major triad $\underline{g}^{-1}$ to the major triad $\underline{c}^{-1}$

5 Semantic Logic in Music Thinking and Its Semantology

A basic question is how to support our understanding of the Mathematical in music. Since the Mathematical is more abstract than logic which itself is more abstract than music, the study of logic in music may particularly contribute to a better understanding of the Mathematical in music thinking.

It is common sense that humans may be affected by music so that it reaches human feelings, emotions, and thought. Humans can even be deeply moved by music, particularly by its musical senses and meanings which may be represented by semantic structures in music (cf. [25]). Now, such structures could be abstracted to semantic structures in logic. For instance, the chords of a well-tempered piano can be abstracted to a logic structure which represents the possible interactions and relationships between those chords.

The result of all such abstractions has been named by the musicologist C. Dahlhaus *"musical logic"* which he characterized by the compositional, technical and esthetic moments which made the automation of instruments possible. Dahlhaus saw the musical logic closely related to the idea of the "language character" of music. That music is presented as sounding discourse, as development of musical thought, is the justification of its esthetic claim, that music is there to be heard for the sake of itself (see [3], p.105f). The richness of this understanding of musical logic is an important assumption for a better understanding of the Mathematical in music thinking.

To obtain even more insights into the Mathematical in music thinking, a further development of the recently introduced *"semantology of music"* could be helpful (cf. [25]). In particular, its philosophic-logical level is basic for the analysis of the Mathematical because philosophical concepts with their objects, their attributes, and their relationships" are highly abstract, but still deduced from actual realities (cf. [10], [5]). The supportive mathematical level is already elaborated to a great extent by methods of *Formal Concept Analysis* (see [8], [9], [24]).

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