The Pac Logic in the properties of C_ω and C_{min}

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Abstract. In this work we try to answer some questions related to the theory of paraconsistent logics. We study a chain of paraconsistent logics stronger than C_ω.

Key words: Paraconsistency, C-systems, non-triviality and non-explosiveness

1 Introduction

When proposing the first paraconsistent propositional calculus, Jaskowski expected it to enjoy the following properties: a) When applied to inconsistent systems, it should not always entail their trivialization; b)It should be rich enough to enable practical inference and c)It should have an intuitive justification. In 1963, da Costa [1] proposed a whole hierarchy of paraconsistent propositional calculi, known as C_n, with $1 \leq n < \omega$: in this calculi, the principle of non-contradiction must not be a valid schema. This lattice of paraconsistent logics will be our study object.

2 Paraconsistent logics

In [1], da Costa suggests the study of non-trivial contradictory logics, which he called paraconsistent logics. A logic L is paraconsistent if $\exists \Gamma : \exists A : \exists B : (\Gamma \vdash A, \Gamma \vdash \neg A$ y $\Gamma \nvdash B)$.

The Pac logic. This logic is of particular importance in our study, it is determined by Table 1, and the connectives $\vee$ and $\wedge$ are determined by the functions *max* and *min*, respectively.

$\longrightarrow$	0	$\frac{1}{2}$	1	$\neg$
0	1	1	1	1
$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1	0

Table 1. Pac's Semantics.

The designated values are 1 and $\frac{1}{2}$. Pac does not accept strong negation, does not admit a bottom particle, is left-adjunctive and is not finitely trivializable.

3 The hierarchy C_n

Let $1 \leq n \leq \omega$. To define C_n, we start with $A^o = \neg(A \wedge \neg A)$ and we write A^n instead of $A^{o \cdots o}$ (n-times). We also write $A^{(n)}$ for $A^1 \wedge A^2 \wedge \cdots \wedge A^n$. It is necessary to clarify that for $n = 1, B^o = B^1 = B^{(1)}$. The only inference rule is Modus Ponens (MP), and the axioms for each C_n are:

- **Pos1.** $A \to (B \to A)$
- **Pos2.** $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
- **Pos3.** $(A \wedge B) \to A$
- **Pos4.** $(A \wedge B) \to B$
- **Pos5.** $A \to (B \to A \wedge B)$
- **Pos6.** $A \to (A \vee B))$
- **Pos7.** $B \to (A \vee B))$
- **Pos8.** $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$
- **C$_\omega$1.** $A \vee \neg A$
- **C$_\omega$2.** $\neg\neg A \to A$
- **12-n.** $B^{(n)} \to (\,(A \to B) \to ((A \to \neg B) \to \neg A)\,)$
- **13-n.** $A^{(n)} \wedge B^{(n)} \to (A \wedge B)^{(n)}$
- **14-n.** $A^{(n)} \wedge B^{(n)} \to (A \vee B)^{(n)}$
- **15-n.** $A^{(n)} \wedge B^{(n)} \to (A \to B)^{(n)}$

The C_ω logic. It is built with the same axioms of C_n, except 12-n to 15-n. Let us denote by C_0 the classical propositional calculus. Then C_n, with $0 \leq n < \omega$, is finitely trivializable. C_ω is not finitely trivializable.

One important results of Arruda is that for $1 \leq n \leq \omega$, it is impossible to reduce the negation. In other words, for $m \neq k$, the following schemes are not valid in C_n (where $\neg_n A$ represents $\neg\neg\cdots\neg A$, n-times). $A \equiv \neg_m A, \quad \neg_{2m} A \equiv \neg_{2k} A, \quad \neg_{2m} A \equiv \neg_{2k+1} A, \quad \neg_{2m+1} A \equiv \neg_{2k+1} A$..

It is important to point out that except for C_0, the calculi C_n are not decidable by using finite matrices. In fact there are valuations (not satisfying the principle of functional truth) that let us prove the soundness of each C_n.

3.1 Decidability of C_n

In [4] da Costa defines valuations for each C_n those valuations do not satisfy the principle of functional truth, then we can not determine thorough valuations whether a formula is valid or not. It is not possible in general to associate a matrix to a formula so we will use the concept of quasi-matrix to refer to an array that differs from a matrix in the following way: A quasi-matrix can show bifurcations in a row starting at some column, the last column is reserved to represent the principal formula, the remaining columns represent proper subformulas and bifurcations show up due to the presence of the connective $\neg$.

Some important results are that: For every line k in the quasi-matrix M, there exists a valuation v such that v_Γ corresponds to k, where Γ is the set of formulas in M and that C_1 is decidable through the valuation v. In order to construct quasi-matrices, it is necessary the following observation, which is characteristic of them:
$v_n(\neg(B^{n-1} \wedge \neg B^{n-1})) = v_n(\neg(\neg B^{n-1} \wedge B^{n-1})) = 0$. Also for $1 \leq n < \omega$. Then C_n is deducible through quasi-matrices. □ and for $0 \leq n \leq \omega$ each of the calculi in the hierarchy C_n is strictly stronger than its successor.

Due to the previous result, we have a family of strictly decreasing Paraconsistent logics which are finitely trivializable due to the fact that they accept strong negation; therefore, they have the bottom particle. Also these results motivates to consider C_ω was a syntactic limit of C_n. Let us remember that C_ω is not finitely trivializable, and can not be finitely gently explosive. We will keep exploding the idea of regarding C_ω as a syntactic limit in order to get more properties.

The C_{min} logic. It is the logic defined when adding the formula $A \vee (A \rightarrow B)$ as an axiom to C_ω. Using a Similar valuation to those of C_n we have soundness and completeness for C_{min}.

Theorem 1. *The calculus C_{min} is not decidable through finite matrices, does not have a bottom particle, it is not finitely trivializable and it does not accept strong negation.*

The proof of this result is a consequence of the semantics proposed by Arruda and the fact that C_{min} is sound under the matrices of Pac logic, and Pac does not accept a bottom particle. With the previous theorem we can realize that C_{min} is not the syntactic limit of C_n, however is seems to be the syntactic closure of C_ω .

4 Conclusions

Pacs semantic makes the C_ω and C_{min} logics sound, as a consequence these logics do not have a bottom particle. It is important to notice that the Arruda's proposal to attack the same problem is much more complicated. The logic C_{min} has come to substitute C_ω as the syntactic limit of the hierarchy C_n.

References

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