Paraconsistent Reasoning for Expressive and Tractable Description Logics ⋆

Yue Ma[1], Pascal Hitzler[2], and Zuoquan Lin[1]

[1]Department of Information Science, Peking University, China
[2]AIFB, Universität Karlsruhe, Germany
{mayue,lz}@is.pku.edu.cn, hitzler@aifb.uni-karlsruhe.de

Abstract. Four-valued description logic has been proposed to reason with description logic based inconsistent knowledge bases, mainly $\mathcal{ALC}$. This approach has a distinct advantage that it can be implemented by invoking classical reasoners to keep the same complexity as classical semantics. In this paper, we further study how to extend the four-valued semantics to more expressive description logics, such as $\mathcal{SHIQ}$, and to more tractable description logics including $\mathcal{EL}$++, DL-Lite, and Horn-DLs. The most effort we spend defining the four-valued semantics of expressive four-valued description logics is on keeping the reduction from four-valued semantics to classical semantics as in the case of $\mathcal{ALC}$; While for tractable description logics, we mainly focus on how to maintain their tractability when adopting four-valued semantics.

1 Introduction

Expressive and tractable description logics have been well-studied in the field of semantic web applications [12, 13]. However, real knowledge bases and data for Semantic Web applications will rarely be perfect. They will be distributed and multi-authored. They will be assembled from different sources and reused. It is unreasonable to expect such realistic knowledge bases to be always logically consistent, and it is therefore important to study ways of dealing with inconsistencies in both expressive and tractable description logic based ontologies, as classical description logics break down in the presence of inconsistent knowledge.

About inconsistency handling of ontologies based on description logics, two fundamentally different approaches can be distinguished. The first is based on the assumption that inconsistencies indicate erroneous data which is to be repaired in order to obtain a consistent knowledge base, e.g. by selecting consistent subsets for the reasoning process [14, 5, 4]. The other approach yields to the insight that inconsistencies are a natural phenomenon in realistic data which are to be handled by a logic which tolerates it [11, 15, 8]. Such logics are called paraconsistent, and the most prominent of them are based on the use of additional truth values standing for *underdefined* (i.e. neither true

⋆ We acknowledge support by the German Federal Ministry of Education and Research (BMBF) under the SmartWeb project (grant 01 IMD01 B), by the EU under the IST project NeOn (IST-2006-027595, http://www.neon-project.org/), and by the Deutsche Forschungsgemeinschaft (DFG) in the ReaSem project.

nor false) and *overdefined* (or *contradictory*, i.e. both true and false). Such logics are appropriately called *four-valued logics* [2, 1]. We believe that either of the approaches is useful, depending on the application scenario. Besides this, four-valued semantics proves useful for measuring inconsistency of ontologies [9], which can provide context information for facilitating inconsistency handling.

In this paper, based on our study of paraconsistent semantics for $\mathcal{ALC}$ in [8], we contribute to the inconsistency handling for DLs in terms of the four-valued semantics for expressive and tractable DLs in following aspects:

- The extension of four-valued semantics to $\mathcal{SHIQ}$ is defined. Specially, we show that it still can be reduced to classical semantics regardless its high expressivity.
- The extension of four-valued semantics to tractable description logics $\mathcal{EL}$++, Horn-DLs, DL-Lite family are studied one by one. We show that the internal inclusion axiom form is a safe way to maintain the tractability when adopting four-valued semantics.
- Compared with our existing work on four-valued semantics of $\mathcal{ALC}$, in this paper, we do not impose four-valued semantics on roles for DLs except DL-Lite. The reasons are: 1) Negative roles are not used as concept constructors in $\mathcal{ALC}$, $\mathcal{SHIQ}$, $\mathcal{EL}$++, or Horn-DLs such that contradiction caused directly by roles can be ignored. 2) We claim that the four-valued semantics should be defined as classically as possible. 3) Four-valued semantics is semantically weaker than classical semantics (the syllogism does not hold under four-valued entailment). So if we adopt four-valued semantics for roles, then we have $\{R \sqsubseteq S, R(a,b)\} \not\models_4 S(a.b)$ even though there is no contradiction in the precondition.

The paper is structured as follows. We first review briefly the four-valued semantics for $\mathcal{ALC}$ in Section 2. Then we study the four-valued semantics for expressive description logics in Section 3 and four-valued semantics for tractable description logics in Section 4, respectively. We conclude and discuss future work in Section 5.

2 Preliminaries

2.1 The Four-valued Semantics for $\mathcal{ALC}$

We describe the syntax and semantics of four-valued description logic $\mathcal{ALC}4$ [8]. Syntactically, $\mathcal{ALC}4$ hardly differs from $\mathcal{ALC}$. Complex concepts and assertions are defined in exactly the same way. For class inclusion, however, significant effort has been devoted on the intuitions behind these different implications in [8]. We claim that various inclusion axioms provide flexible ways to model inconsistent ontologies. They are as follows:

$$C \mapsto D \text{ (material inclusion axiom)},$$
$$C \sqsubset D \text{ (internal inclusion axiom)},$$
$$C \rightarrow D \text{ (strong inclusion axiom)}.$$

Semantically, interpretations map individuals to elements of the domain of the interpretation, as usual. For concepts, however, modifications are made to the notion of interpretation in order to allow for reasoning with inconsistencies.

Table 1. Semantics of $\mathcal{ALC}4$ Concepts

Constructor Syntax	Semantics
A	$A^I = \langle P, N \rangle$, where $P, N \subseteq \Delta^I$
o	$o^I \in \Delta^I$
$\top$	$\langle \Delta^I, \emptyset \rangle$
$\bot$	$\langle \emptyset, \Delta^I \rangle$
$C_1 \sqcap C_2$	$\langle P_1 \cap P_2, N_1 \cup N_2 \rangle$, if $C_i^I = \langle P_i, N_i \rangle$ for $i = 1, 2$
$C_1 \sqcup C_2$	$\langle P_1 \cup P_2, N_1 \cap N_2 \rangle$, if $C_i^I = \langle P_i, N_i \rangle$ for $i = 1, 2$
$\neg C$	$(\neg C)^I = \langle N, P \rangle$, if $C^I = \langle P, N \rangle$
$\exists R.C$	$\langle \{x \mid \exists y, (x, y) \in R^I$ and $y \in proj^+(C^I)\},$ $\{x \mid \forall y, (x, y) \in R^I$ implies $y \in proj^-(C^I)\}\rangle$
$\forall R.C$	$\langle \{x \mid \forall y, (x, y) \in R^I$ implies $y \in proj^+(C^I)\},$ $\{x \mid \exists y, (x, y) \in R^I$ and $y \in proj^-(C^I)\}\rangle$

Intuitively, in four-valued logic we need to consider four situations which can occur in terms of containment of an individual in a concept: (1) we know it is contained, (2) we know it is not contained, (3) we have no knowledge whether or not the individual is contained, (4) we have contradictory information, namely that the individual is both contained in the concept and not contained in the concept. There are several equivalent ways how this intuition can be formalised, one of which is described in the following.

For a given domain Δ^I and a concept C, an interpretation over Δ^I assigns to C a pair $\langle P, N \rangle$ of (not necessarily disjoint) subsets of Δ^I. Intuitively, P is the set of elements known to belong to the extension of C, while N is the set of elements known to be not contained in the extension of C. For simplicity of notation, we define functions $proj^+(\cdot)$ and $proj^-(\cdot)$ by $proj^+\langle P, N \rangle = P$ and $proj^-\langle P, N \rangle = N$.

Formally, a four-valued interpretation is a pair $I = (\Delta^I, \cdot^I)$ with Δ^I as domain, where $\cdot^I$ is a function assigning elements of Δ^I to individuals, and subsets of $(\Delta^I)^2$ to concepts, such that the conditions in Table 1 are satisfied. Note that the semantics of roles here remains unchanged from the classical two-valued case. Intuitively, inconsistencies always rise on concepts, and not on roles, at least in the absence of role negation, which is often assumed when studying DLs. We will see in this paper that this approach can be used to tolerate inconsistency, not only for $\mathcal{ALC}$ but also for more expressive description logics. This is an improvement over [8] in the sense that we would like to make as few changes as possible when extending the classical semantics to a four-valued semantics for handling inconsistency.

The semantics of the three different types of inclusion axioms is formally defined in Table 2 (together with the semantics of concept assertions). we refer to [8] for details.

We say that a four-valued interpretation I satisfies a four-valued knowledge base $\mathcal{O}$ (i.e. is a model of it) iff it satisfies each assertion and each inclusion axiom in $\mathcal{O}$. A knowledge base $\mathcal{O}$ is satisfiable (unsatisfiable) iff there exists (does not exist) such a model.

Table 2. Semantics of inclusion axioms in $\mathcal{ALC}4$

Axiom Name	Syntax	Semantics
material inclusion	$C_1 \mapsto C_2$	$\Delta^I \setminus proj^-(C_1^I) \subseteq proj^+(C_2^I)$
internal inclusion	$C_1 \sqsubset C_2$	$proj^+(C_1^I) \subseteq proj^+(C_2^I)$
strong inclusion	$C_1 \rightarrow C_2$	$proj^+(C_1^I) \subseteq proj^+(C_2^I)$ and $proj^-(C_2^I) \subseteq proj^-(C_1^I)$
individual assertions	$C(a)$	$a^I \in proj^+(C^I)$
	$R(a,b)$	$(a^I, b^I) \in R^I$

2.2 Reduction from Four-valued Semantics of $\mathcal{ALC}$ to Classical Semantics

It is a pleasing property of $\mathcal{ALC}4$ that it can be translated easily into classical $\mathcal{ALC}$, such that paraconsistent reasoning can be simulated by using standard $\mathcal{ALC}$ reasoning algorithms.

Definition 1 *(Concept transformation) For any given concept C, its transformation $\pi(C)$ is the concept obtained from C by the following inductively defined transformation.*

- *If $C = A$ for A an atomic concept, then $\pi(C) = A^+$, where A^+ is a new concept;*
- *If $C = \neg A$ for A an atomic concept, then $\pi(C) = A'$, where A' is a new concept;*
- *If $C = \top$, then $\pi(C) = \top$;*
- *If $C = \bot$, then $\pi(C) = \bot$;*
- *If $C = E \sqcap D$ for concepts D, E, then $\pi(C) = \pi(E) \sqcap \pi(D)$;*
- *If $C = E \sqcup D$ for concepts D, E, then $\pi(C) = \pi(E) \sqcup \pi(D)$;*
- *If $C = \exists R.D$ for D a concept and R is a role, then $\pi(C) = \exists R.\pi(D)$;*
- *If $C = \forall R.D$ for D a concept and R is a role, then $\pi(C) = \forall R.\pi(D)$;*
- *If $C = \neg\neg D$ for a concept D, then $\pi(C) = \pi(D)$;*
- *If $C = \neg(E \sqcap D)$ for concepts D, E, then $\pi(C) = \pi(\neg E) \sqcup \pi(\neg D)$;*
- *If $C = \neg(E \sqcup D)$ for concepts D, E, then $\pi(C) = \pi(\neg E) \sqcap \pi(\neg D)$;*
- *If $C = \neg(\exists R.D)$ for D a concept and R is a role, then $\pi(C) = \forall R.\pi(\neg D)$;*
- *If $C = \neg(\forall R.D)$ for D a concept and R is a role, then $\pi(C) = \exists R.\pi(\neg D)$;*

Based on this, axioms are transformed as follows.

Definition 2 *(Axiom Transformations) For any ontology O, $\pi(O)$ is defined as the set $\{\pi(\alpha) \mid \alpha$ is an axiom of $O\}$, where $\pi(\alpha)$ is the transformation performed on each axiom defined as follows:*

- $\pi(\alpha) = \neg\pi(\neg C_1) \sqsubseteq \pi(C_2)$, *if* $\alpha = C_1 \mapsto C_2$;
- $\pi(\alpha) = \pi(C_1) \sqsubseteq \pi(C_2)$, *if* $\alpha = C_1 \sqsubset C_2$;
- $\pi(\alpha) = \{\pi(C_1) \sqsubseteq \pi(C_2), \pi(\neg C_2) \sqsubseteq \pi(\neg C_1)\}$, *if* $\alpha = C_1 \rightarrow C_2$;.
- $\pi(C(a)) = \pi(C)(a), \pi(R)(a,b) = R(a,b)$,

where a, b are individuals, C_1, C_2, C are concepts, R a role.

We note two issues. First of all, the transformation algorithm is linear in the size of the ontology. Secondly, for any $\mathcal{ALC}$ ontology O, $\pi(O)$ is still an $\mathcal{ALC}$ ontology. Based on these two observations as well as the following theorem, we can see that paraconsistent reasoning of $\mathcal{ALC}$ can indeed be simulated on standard reasoners by means of the transformation just given.

Theorem 1 *For any ontology O in $\mathcal{ALC}$, O is 4-valued unsatisfiable if and only if $\pi(O)$ is unsatisfiable under the classical semantics of $\mathcal{ALC}$.*

Definition 3 *Given a knowledge base O, the satisfiable form of O, written $SF(O)$, is a knowledge base obtained by replacing each occurrence of $\bot$ in O with $A_{new} \sqcap \neg A_{new}$, and replacing each occurrence of $\top$ in (O) with with $A_{new} \sqcup \neg A_{new}$, where A_{new} is a new atomic concept.*

3 Paraconsistent Semantics for Expressive DLs

In this section, we study how to extend four-valued semantics to $\mathcal{SHIQ}$.

For the conflicting assertion set $\{\geq (n+1)R.C(a), \leq nR.C(a)\}$, intuitively, it is caused by the contradiction that there should be less than n different individuals related to a via the R relation, and also there should be more than $n+1$ different individuals related to a via R. That is, the contradiction is from the set of individuals of concept C which relate a via R. By this idea, we extend the four-valued semantics to the constructors for number restrictions in Table 3. We remark that the semantics of roles is just the classical semantics. So the semantics for role inclusion and transitive role axiom are still classical.

Table 3. Four-valued Semantics Extension to Number Restrictions and Nominals

Constructor	Semantics
$\geq nR.C$	$\langle\{x \mid \#(y.(x,y) \in R^I \wedge y \in proj^+(C^I)) \geq n\}$,
	$\{x \mid \#(y.(x,y) \in R^I \wedge y \notin proj^-(C^I)) < n\}\rangle$
$\leq nR.C$	$\langle\{x \mid \#(y.(x,y) \in R^I \wedge y \notin proj^-(C^I)) \leq n\}$,
	$\{x \mid \#(y.(x,y) \in R^I \wedge y \in proj^+(C^I)) > n\}\rangle$

Example 1 *Consider $\{\geq 2hasStu.PhD(Green), \leq 1hasStu.PhD(Green)\}$ which says the conflicting facts that Green has at least two and at most one PhD student. Consider a 4-interpretation: $I = (\Delta^I, \cdot^I)$ where $\Delta^I = \{a_1, a_2, b_1, b_2, Green\}$, $PhD^I = \langle\{a_1, b_1\}, \{b_1, b_2, a_2\}\rangle$, and $hasStu^I = \{(Green, a_1), (Green, a_2), (Green, b_1), (Green, b_2)\}$. According to Table 3, we can see that I is a 4-model because $(\geq 2hasStu.PhD(Green))^I = (\leq 1hasStu.PhD(Green))^I = B$ by checking*

$$Green \in \{x \mid \#(y.(x,y) \in hasStu^I \wedge y \in proj^+(PhD^I)) \geq 2\},$$
$$Green \in \{x \mid \#(y.(x,y) \in hasStu^I \wedge y \notin proj^-(PhD^I)) < 2\}.$$

That is, the conflicting assertions are assigned the contradictory truth value B under their 4-model I.

For the extended four-valued semantics defined in Table 3, we have following properties hold as under classical semantics.

Proposition 2 *Let C be a concept and R be an object role name. For any four-valued interpretation I defined satisfying Table 3, we have*

$$(\neg(\leq nR.C))^I =_4 (> nR.C)^I \quad \textit{and} \quad (\neg(\geq nR.C))^I =_4 (< nR.C)^I.$$

Proposition 3 *Let C be a concept and R be an object role name. For any four-valued interpretation I defined satisfying Table 3, we have*

$$(\exists R.C)^I =_4 (\geq 1R.C)^I \quad \textit{and} \quad (\forall R.C)^I =_4 (< 1R.\neg C)^I.$$

Proposition 2 and Proposition 3 show that many intuitive relations between different concept constructors still hold under the four-valued semantics, which is one of nice properties of our four-valued semantics for handling inconsistency.

Next proposition shows that our definition of four-valued semantics for $\mathcal{SHIQ}$ is enough to handle inconsistencies in an $\mathcal{SHIQ}$ knowledge base.

Proposition 4 *For any $\mathcal{SHIQ}$ knowledge base O, SF(O) always has at least one 4-valued model, where SF$(\cdot)$ operator is defined in Definition 3.*

Note that unqualified number restrictions, $\geq n.R$ and $\leq n.R$ are special forms of number restrictions because of the equations $\leq n.R =_2 \leq nR.\top$ and $\geq n.R =_2 \geq nR.\top$. However, if we defined the four-valued semantics of $\leq n.R(\geq n.R)$ by the four-valued semantics of $\leq nR.\top(\geq nR.\top)$ defined in Table 3 and Table 1, we would find that $\{\leq n.R(a), \geq n{+}1.R(a)\}$ is still an unsatisfiable set. This is because $\#(y.(a,y) \in \textit{proj}(R^I) \wedge y \in \textit{proj}^+(\top^I)) \geq n+1$ and $\#(y.(a,y) \in \textit{proj}(R^I) \wedge y \notin \textit{proj}^-(\top^I)) \leq n$ cannot hold simultaneously since $\top^I = \langle \Delta^I, \emptyset \rangle$.

To address this problem, we also adopt the *substitution* defined by Definition 3. By substituting $\top$ by $A_{new} \sqcup \neg A_{new}$ in $\geq (n+1)R.\top$ and $\leq nR.\top$, we can see that $\{\leq n.R(a), \geq n+1.R(a)\}$ has a four-valued model with $\Delta^I = \{a, b_1, ..., b_{n+1}\}, (a, b_i) \in R^I$ for $1 \leq i \leq n+1$, *and* $A^I_{new} = \langle \Delta^I, \Delta^I \rangle$. By doing this, we get a four-valued model I which pushes the contraction onto the new atomic concept A_{new}.

Next we study how to extend the reduction algorithm to the case of four-valued semantics of $\mathcal{SHIQ}$.

Definition 4 *(Definition 1 extended) For any given concept C, its transformation $\pi(C)$ is the concept obtained from C by the following inductively defined transformation.*

- *If $C = \geq nR.D$ for D a concept and R a role, then $\pi(C) = \geq nR.\pi(D)$;*
- *If $C = \leq nR.D$ for D a concept and R a role, then $\pi(C) = \leq nR.\neg\pi(\neg D)$;*
- *If $C = \neg(\geq nR.D)$ for D a concept and R a role, then $\pi(C) = < nR.\neg\pi(\neg D)$;*
- *If $C = \neg(\leq nR.D)$ for D a concept and R a role, then $\pi(C) = > nR.\pi(D)$;*

Regarding both the extension of number restrictions and of nominals, the following theorem holds, which lays the theoretical foundation for the algorithm of four-valued semantics for expressive DLs.

Theorem 5 *(Theorem 1 extended) For any ontology O in $\mathcal{SHIQ}$, O is 4-valued unsatisfiable if and only if $\pi(O)$ is unsatisfiable under the classical semantics of $\mathcal{SHIQ}$.*

4 Tractable DLs

In this section, we will see that inconsistencies are also unavoidable in many tractable DLs. So we focus on discussing whether the four-valued semantics can preserve the tractability of these tractable DLs. That is, whether the reduction for computing the four-valued semantics transfers tractable DLs still into tractable DLs. If it does, then we can use the four-valued semantics to deal with inconsistency without having to worry about intractability. Our discussion is based on $\mathcal{EL}$++, Horn-DLs, and DL-Lite.

4.1 $\mathcal{EL}$++

We do not consider concrete domains. The syntax definition of $\mathcal{EL}$++ knowledge bases is shown in Table 4. $\mathcal{EL}$++ ontologies may also contain role inclusions (RI) of the form $r_1 \circ \cdots \circ r_k \sqsubseteq r$, where $\circ$ denotes role composition.

It is easy to see that an $\mathcal{EL}$++ knowledge base may be inconsistent if we consider the knowledge base $\{A \sqsubseteq \bot, A(a)\}$. So we still hope that the 4-valued semantics can help us to handle inconsistency in $\mathcal{EL}$++ knowledge bases. However, we will see that we don't have as many choices of class inclusion as in $\mathcal{ALC}$ and $\mathcal{SHIQ}$ if we want to maintain the tractability of the 4-valued entailment relationship of $\mathcal{EL}$++. The analysis is as follows.

Obviously, the concept transformation of Definition 1 performing on an $\mathcal{EL}$++ concept produces an $\mathcal{EL}$++ concept. For the transformation of internal inclusion, each $\mathcal{EL}$++ axiom $C \sqsubset D$ is transformed into $\pi(C) \sqsubseteq \pi(D)$ where $\pi(C)$ and $\pi(D)$ are still $\mathcal{EL}$++ concepts, so that $\pi(C) \sqsubseteq \pi(D)$ is still an $\mathcal{EL}$++ axiom. So internal class inclusion does not destroy the tractability of $\mathcal{EL}$++. This property does not hold for material and strong class inclusions as shown by the following counterexamples: $A \sqcap A' \mapsto B$ and $A \sqcap A' \rightarrow B$. They will be transformed into $\neg(A^- \sqcup A'^-) \sqsubseteq B^+$ and $\{A^+ \sqcap A'^+ \sqsubseteq B^+, B^- \sqsubseteq (A^- \sqcup A'^-)\}$ by Definition 2, which are not within the expressivity of $\mathcal{EL}$++. This is mainly because of no negative constructor in $\mathcal{EL}$++.

For role inclusions in $\mathcal{EL}$++, since there is no negative role constructor which can cause inconsistency, we only need to use the classical interpretation for roles as what we do for $\mathcal{ALC}$. So adaptation of 4-valued semantics does not effect the role inclusions axioms.

4.2 Horn-DLs

We ground our discussion on Horn-$\mathcal{SHOIQ}\circ$ as defined in [7]. Then we will point out that the same conclusion holds for other Horn-DLs, like Horn-$\mathcal{SHIQ}$ [10], which has

Table 4. $\mathcal{EL}$++ and Horn-$\mathcal{SHOIQ}\circ$. The Horn-$\mathcal{SHOIQ}\circ$ normal form used is due to [7].

Language	GCIs	Tractability-preserving Inclusions
$\mathcal{EL}++$	$C \sqsubseteq D$, where $C, D = \top \mid \bot \mid \{a\} \mid C_1 \sqcap C_2 \mid \exists r.C$	internal inclusion (only)
Horn-$\mathcal{SHOIQ}\circ$	$\top \sqsubseteq A$, $A \sqsubseteq \bot$, $A \sqcap A' \sqsubseteq B$, $\exists R.A \sqsubseteq B$, $A \sqsubseteq \exists R.B$, $A \sqsubseteq \forall S.B$, $A \sqsubseteq \geq nR.B$, $A \sqsubseteq \leq 1R.B$.	internal inclusion (only)

tractable data complexity [6]. We define Horn-$\mathcal{SHOIQ}\circ$ by means of a normal form given in [7], which can be found in Table 4 where A, A', B are concept names.

We can see that all of the Horn-$\mathcal{SHOIQ}\circ$ concept constructors preserve its form under $\pi(\cdot)$ operator except $\leq 1R.B$, because $\pi(\leq 1R.B) =\leq 1R.\neg B^-$ according to Definition 4. To still maintain the concept structure of $\leq 1R.B$ within Horn-$\mathcal{SHOIQ}\circ$, we redefine the $\pi(\cdot)$ operator on $\leq 1R.C$ as

$$\pi(\leq 1R.C) = \{\leq 1R.C^=\}, \text{ where } C^= \sqcap C^- \sqsubseteq \bot.$$

Then $\pi(A \sqsubset \leq 1R.C) = \{A \sqsubseteq \leq 1R.C^=, C^= \sqcap C^- \sqsubseteq \bot\}$ by the transformation definition of internal inclusion axiom.

However, not all of the axiom transformations in Definition 2 preserve the structures of Horn-DL axioms. As the analysis in last paragraph, internal inclusion preserves Horn-ness. As counterexamples for material inclusion and strong inclusion, just consider again the counterexample used in $\mathcal{EL}$++ case. The transformed forms $\neg(A^- \sqcup A'^-) \sqsubseteq B^+$ and $\{A^+ \sqcap A'^+ \sqsubseteq B^+, B^- \sqsubseteq (A^- \sqcup A'^-)\}$ are not within the expressivity of Horn-$\mathcal{SHOIQ}\circ$. Since $A \sqcap A' \sqsubseteq B$ is allowed in other Horn-DLs, the same conclusion as for Horn-$\mathcal{SHOIQ}\circ$ holds. This means that when we want ro preserve the structure of tractable Horn-DLs, we have to choose internal inclusion as the only inclusion form to perform paraconsistent reasoning.

4.3 DL-Lite

DL-Lite family includes DL-Lite$_{core}$, DL-Lite$_{\mathcal{F}}$, and DL-Lite$_{\mathcal{R}}$. The logics of DL-Lite family are the maximal DLs supporting efficient query answering over large amounts of instances. In [3], the usual DL reasoning tasks on DL-Lite family are shown to be polynomial in the size of the TBox, and query answering is LogSpace in the size of the ABox. Moreover, DL-Lite family allows for separation between TBox and ABox reasoning during query evaluation: the part of the process requiring TBox reasoning is independent of the ABox, and the part of the process requiring to the ABox can be carried out by an SQL engine [3].

Concepts and roles of DL-Lite family are formed by the following syntax [3]:

$$B ::= A \mid \exists R \qquad R ::= P \mid P^-$$
$$C ::= B \mid \neg B \qquad E ::= R \mid \neg R$$

where A denotes an atomic concept, P an atomic role, and P^- the inverse of the atomic role P. See to Table 5 for the syntax definitions of GCIs and Role Inclusions.

Table 5. DL-Lite Family

Language	GCIs	Role Inclusions	Tractability-preserving Inclusions
DL-Lite$_{core}$	$B \sqsubseteq C$	$\emptyset$	internal inclusion (only)
DL-Lite$_{\mathcal{R}}$	$B \sqsubseteq C$	$R \sqsubseteq E$	internal inclusion (only)
DL-Lite$_{\mathcal{F}}$	$B \sqsubseteq C$	(funct R)	internal inclusion (only)

It is also easy to construct an inconsistent knowledge base even for DL-Lite$_{core}$. For instance, *KB* $= \{B \sqsubseteq \neg A, B(a), A(a)\}$. Moreover, conflictions about roles possibly occur on DL-Lite$_{\mathcal{R}}$, such as $\{P \sqsubseteq P', P \sqsubseteq \neg P', P(a, b)\}$.

In order to still adopt 4-valued semantics for DL-Lite family, we define the four-valued semantics extension for roles. Just as the four-valued semantics for concepts, a pair $\langle R_P, R_N \rangle$ ($R_P, R_N \subseteq (\Delta^I)^2$) denotes the four-valued semantics of a role R under interpretation I, where R_P stands for the set of pairs of individuals which are related via R and R_N explicitly represents the set of pairs of individuals which are not related via R. Table 6 gives the formal definition.

Table 6. Four-valued Semantics of Roles

Syntax about Roles	**Semantics**
R	$R^I = \langle R_P, R_N \rangle$, where $R_P, R_N \subseteq \Delta^I \times \Delta^I$
R^-	$(R^-)^I = \langle R_P^-, R_N^- \rangle$, where R_P^-, R_N^- represent the inverse relations on R_P^- and R_N^-, respectively.
$\exists R$	$\langle \{x \mid \exists y, (x, y) \in R_P^I\}, \{x \mid \forall y, (x, y) \in R_N^I\} \rangle$
$\neg\exists R$	$\langle \{x \mid \forall y, (x, y) \in R_N^I\}, \{x \mid \exists y, (x, y) \in R_P^I\} \rangle$

For simple expression, we say that x and y are *positively related* via R under interpretation I if $(x, y) \in R_P^I$, and that x and y are *negatively related* via R under interpretation I if $(x, y) \in R_N^I$.

Intuitively, the first part of the four-valued semantics $\exists R$ in Table 6 denotes the set of individuals x which has an individual y positively related x via R. While the second part denotes the set of individuals x which negatively relate with all individuals y via R. Note that x is not negatively related to y does not mean x and y are positively related under four-valued semantics, since $R_P^I \cup R_N^I = \Delta^I \times \Delta^I$ and $R_P^I \cap R_N^I = \emptyset$ are not necessary to hold under four-valued semantics. This also the key point why our four-valued semantics can tolerant conflicts caused by role assertions, by allowing a, b both positively related and negatively related via R under a four-valued interpretation I. We give the following example to illustrate the four-valued semantics of $\exists R$.

Example 2 *Note that the ontology* $O = \{\exists hasStud(Green), \neg\exists hasStud(Green)\}$ *is inconsistent. Consider the following 4-interpretation* $I = (\Delta^I, \cdot^I)$*, where* $\Delta^I = \{a, b, Green\}$*:*

$$hasStud^I = \langle \{(Green, a)\}, \{(Green, a), (Green, b), (Green, Green)\} \rangle$$

which says that we know that $Green$ *has a student* a*, and that* $Green$ *doesn't relate any of the individuals via the relation* $hasStudent$*. By checking the following formulae we know that* I *is a 4-model of* O*:*

$$Green \in \{\text{three exists } y \in \Delta^I, \text{ such that } (Green, y) \in hasStud_P^I\}$$
$$Green \in \{\text{for all } y \in \Delta^I, (Green, y) \in hasStud_N^I\}.$$

Intuitively, this 4-model reflects the contradictory status of the ontology about whether $Green$ *has a student.*

Now we turn to define the concept transformations for DL-Lite.

Definition 5 *The concept and role transformations for DL-Lite concepts are defined on structure induction as follows.*

- *For* $E = R$*, then* $\pi(E) = R$*;*
- *For* $E = \neg R$*, then* $\pi(E) = R'$*, where* R' *is a new role;*
- *If* $C = \exists R$*, then* $\pi(C) = \exists R$*;*
- *If* $C = \neg\exists R$*, then* $\pi(C) = \neg\exists R^=$*, where* $R^=$ *is a new role name and* $R^= \sqsubseteq \neg R'$*.*

Consider the internal inclusion transformation, we have all the GCIs $B \sqsubseteq C$ of DL-Lite will be transferred into form $B \sqsubseteq C$ with at most an additional role inclusion because $\pi(B \sqsubseteq \neg\exists R) = \{B \sqsubseteq \neg\exists R^=, R^= \sqsubseteq \neg R'\}$. For material inclusion and strong inclusion, because the negative concept is not allowed to occur on the left of a GCI, they do not preserve the DL-Lite structure. So only internal inclusion works under the reduction from four-valued semantics to classical semantics of DL-Lite family to keep tractability. We can see that the four-valued semantics of DL-Lites family can be reduced to the reasoning of classical DL-Lite$_{\mathcal{R}}$. The following theorem shows this.

Theorem 6 *For any DL-Lite ontology* O*,* O *is 4-valued unsatisfiable if and only if* $\pi(O)$ *is unsatisfiable under the classical semantics of DL-Lite*$_{\mathcal{R}}$*.*

5 Conclusions

In this paper, we further studied the four-valued semantics for description logics, specially for expressive DLs and tractable DLs. We formally defined their four-valued semantics and proper reductions to classical semantics, such that all the benefits from existing reasoners on these DLs can also be achieved by invoking classical reasoners after employing these reduction algorithms in a preprocessing manner. And the size of obtained ontology is linear as the size of original ontology. Unlike the four-valued semantics for $\mathcal{ALC}$ and $\mathcal{SHIQ}$, we showed that in order to preserve the tractability of tractable DLs, only internal class inclusion among the three class inclusion forms is suitable to model class inclusion. Intuitively, this is because that the semantics of material and strong class inclusions needs the ability to represent negation in some complex form, which is not within the expressivity of tractable DLs. In the future, we will implement these extensions on our existing prototype ParOWL[1] to support paraconsistent inconsistency handling on these DLs.

References

1. N. D. Belnap. How a computer should think. *Comtemporary Aspects of Philosophy: Proceedings of the Oxford International Symposium*, pages 30–56, 1977.
2. N. D. Belnap. A useful four-valued logic. *Modern uses of multiple-valued logics*, pages 7–73, 1977.
3. Diego Calvanese, Giuseppe De Giacomo, Domenico Lembo, Maurizio Lenzerini, and Riccardo Rosati. Tractable reasoning and efficient query answering in description logics: The *l-lite* family. *J. Autom. Reasoning*, 39(3):385–429, 2007.
4. Peter Haase, Frank van Harmelen, Zhisheng Huang, Heiner Stuckenschmidt, and York Sure. A framework for handling inconsistency in changing ontologies. In Yolanda Gil, Enrico Motta, V. Richard Benjamins, and Mark A. Musen, editors, *International Semantic Web Conference*, volume 3729 of *Lecture Notes in Computer Science*, pages 353–367. Springer, 2005.
5. Zhisheng Huang, Frank van Harmelen, and Annette ten Teije. Reasoning with inconsistent ontologies. In Leslie Pack Kaelbling and Alessandro Saffiotti, editors, *IJCAI*, pages 454–459. Professional Book Center, 2005.
6. Ullrich Hustadt, Boris Motik, and Ulrike Sattler. Data Complexity of Reasoning in Very Expressive Description Logics. In *Proc. of the 19th Int. Joint Conference on Artificial Intelligence (IJCAI 2005)*, pages 466–471, Edinburgh, UK, July 30 – August 5 2005. Morgan Kaufmann Publishers.
7. Markus Krötzsch, Sebastian Rudolph, and Pascal Hitzler. Complexity boundaries for Horn description logics. In *AAAI*, pages 452–457. AAAI Press, 2007.
8. Yue Ma, Pascal Hitzler, and Zuoquan Lin. Algorithms for paraconsistent reasoning with owl. In Enrico Franconi, Michael Kifer, and Wolfgang May, editors, *ESWC*, volume 4519 of *Lecture Notes in Computer Science*, pages 399–413. Springer, 2007.
9. Yue Ma, Guilin Qi, Pascal Hitzler, and Zuoquan Lin. Measuring inconsistency for description logics based on paraconsistent semantics. In Khaled Mellouli, editor, *ECSQARU*, volume 4724 of *Lecture Notes in Computer Science*, pages 30–41. Springer, 2007.
10. Boris Motik. Reasoning in description logics using resolution and deductive databases. *PhD theis, University Karlsruhe, Germany*, 2006.
11. Peter F. Patel-Schneider. A four-valued semantics for terminological logics. *Artificial Intelligence*, 38:319–351, 1989.
12. Peter F. Patel-Schneider and Horrocks Ian. OWL web ontology language semantics and abstract syntax. *W3C Recommendation*, 10 February, 2004.
13. Peter F. Patel-Schneider and Horrocks Ian. OWL 1.1 web ontology language overview. *W3C Member Submssion*, 19 December, 2006.
14. Stefan Schlobach and Ronald Cornet. Non-standard reasoning services for the debugging of description logic terminologies. In Georg Gottlob and Toby Walsh, editors, *IJCAI*, pages 355–362. Morgan Kaufmann, 2003.
15. Umberto Straccia. A sequent calculus for reasoning in four-valued description logics. In Didier Galmiche, editor, *TABLEAUX*, volume 1227 of *Lecture Notes in Computer Science*, pages 343–357. Springer, 1997.

[1] http://logic.aifb.uni-karlsruhe.de/wiki/Paraconsistent_reasoning