Further Galois Connections between Semimodules over Idempotent Semirings

Francisco J. Valverde-Albacete and Carmen Peláez-Moreno *

Dpto. de Teoría de la Señal y de las Comunicaciones.
Universidad Carlos III de Madrid
Avda. de la Universidad, 30. Leganés 28911. Spain
fva,carmen@tsc.uc3m.es

Abstract. In [14] a generalisation of Formal Concept Analysis was introduced with data mining applications in mind, $\mathcal{K}$-Formal Concept Analysis, where incidences take values in certain kinds of semirings, instead of the standard Boolean carrier set. The construction leading to the pair of dually (order) isomorphic lattices can be further manipulated to obtain the three other types of Galois Connections providing a fuller set of tools to interpret any relations between data. We relate this result to previous descriptions of certain instances of such Galois Connections in qualitative data analysis and provide concrete examples of them related to $\overline{\mathbb{R}}_{\max,+}$-semimodules in quantitative data analysis.

1 Motivation: Lattices related to an Incidence

Data analysis results improve when many different tools are offered to the practitioner. Consider then the *modal operators* ([13], def. 3.8.2; [6]) introduced by a Boolean matrix, $I \in 2^{G\times M}$, over a set of objects, $A \in 2^G$ and, dually, over sets $B \in 2^M$ of attributes operated by the converse relation $I^{\mathrm{t}} \in 2^{M\times G}$ as listed in Table 1. Formal Concept Analysis adepts may recognise the extent and intent polars in the sufficiency operators for a relation, $[[I\ ]]\,(A) = A'$, $[[I^{\mathrm{t}}]]\,(B) = B'$, but also their closure operators, $[[I^{\mathrm{t}}]][[I\ ]](A) = A''$, $[[I\ ]]\,[[I^{\mathrm{t}}]]\,(B) = B''$.

Perhaps less known is that the pairs of operators in the first and second rows of Table 1 define the *neighbourhood lattices*: For a formal context (G, M, I) define the *span* of a set of objects as: $\mathrm{span}(A) := \langle I\rangle(A) = (A)^I_\exists$. This is the set of attributes related to some $g \in A$ [1]. Similarly, define for its dual context (M, G, I^{t}) the *content* of a set of attributes, $\mathrm{content}(B) = [I^{\mathrm{t}}]\,(B) = (B)^\forall_I$, as the set of objects which can be completely described by the attributes in B . Next consider the set $\mathfrak{N}(G, M, I)$ (for Ger. *Nachbar*, neighbour) of *neighbour pairs*, $(A, B) \in \mathfrak{N}(G, M, I)$, such that $\mathrm{span}(A) = (A)^I_\exists = B \Leftrightarrow A = (B)^\forall_I = \mathrm{content}(B)$. Then we can state the:

* This work has been partially supported by a grants from the Spanish Government-Comisión Interministerial de Ciencia y Tecnología project TEC2005-04264/TCM.

[1] The second, operator notation is closer to Galois connection theory as explained below and relates better to normal notation in Formal Concept Analysis.

Table 1. Modal operators over a relation and its converse for sets of objects $A \subseteq G$ and attributes $B \subseteq M$. The misalignment in the first two rows is intentional.

possibility operator over G $\langle I \rangle (A) = \{ m \in M \mid (\exists g \in G)[g \in A \wedge gIm] \}$	necessity operator over M $[I^{\mathrm{t}}] (B) = \{ g \in G \mid (\forall m \in M)[mI^{\mathrm{t}}g \Rightarrow m \in B) \}$
necessity operator over G $[I] (A) = \{ m \in M \mid (\forall g \in G)[gIm \Rightarrow g \in A) \}$	possibility operator over M $\langle I^{\mathrm{t}} \rangle (B) = \{ g \in G \mid (\exists m \in M)[m \in B \wedge mI^{\mathrm{t}}g] \}$
sufficiency operator over G $[[I]] (A) = \{ m \in M \mid (\forall g \in G)[g \in A \Rightarrow gIm) \}$	sufficiency operator over M $[[I^{\mathrm{t}}]] (B) = \{ g \in G \mid (\forall m \in M)[m \in B \Rightarrow mI^{\mathrm{t}}g) \}$
dual sufficiency operator over G $\langle\langle I \rangle\rangle (A) = \{ m \in M \mid (\exists g \in G)[g \notin A \wedge g\not{I}m] \}$	dual sufficiency operator over M $\langle\langle I^{\mathrm{t}} \rangle\rangle (B) = \{ g \in G \mid (\exists m \in M)[m \notin B \wedge m \not{I}^{\mathrm{t}}\ g] \}$

Theorem 1 (Fundamental theorem of Neighbourhood lattices [6]). *The neighbourhood lattice, $\mathfrak{N}(G, M, I)$, is a complete lattice in which infimum and supremum are given by:*

$$\bigwedge_{t \in T} (A_t, B_t) = \left(\bigcap_{t \in T} A_t, \left(\left(\bigcap_{t \in T} B_t \right)^{\forall}_{I} \right)^{I}_{\exists} \right) \quad \bigvee_{t \in T} (A_t, B_t) = \left(\left(\left(\bigcup A_t \right)^{I}_{\exists} \right)^{\forall}_{I}, \bigcup_{t \in T} B_t \right)$$

Conversely, a complete lattice V is isomorphic to $\mathfrak{N}(G, M, I)$ if and only if there are mappings[2] *$\sigma : G \to V$ and $\kappa : M \to V$ such that $\sigma(G) \cup \{0\}$ is join-dense in V and $\kappa(M) \cup \{1\}$ is meet-dense in V and gIm is equivalent to $\sigma(g) \nleq \kappa(m)$ for all $g \in G$ and all $m \in M$.*

This result shows that $\mathfrak{N}(G, M, I)$ is isomorphic to $\underline{\mathfrak{B}}(G, M, \not{I})$ [6], but the drawing and interpretation in terms of neighbourhood pairs must differ, since the systems of spans and contents are now order isomorphic: if (A_1, B_1), (A_2, B_2) are neighbour pairs, they are ordered by the relation: $(A_1, B_1) \leq (A_2, B_2) \iff A_1 \leq A_2 \iff B_1 \leq B_2$.

In a different train of thought and with algebraic applications in mind, when considering finite incidences to be represented by Boolean matrices, $I = [I_{ij}]$ and sets of objets by Boolean (column) vectors, we may write: $\langle I \rangle (A) = \bigvee_j I_{ij} \wedge a_j$, $[I] (A) = \bigwedge_j I_{ij} \to a_j$, $\langle\langle I \rangle\rangle (A) = \bigvee_j \overline{I_{ij}} \wedge \overline{x_j}$ and $[[I]] (A) = \bigwedge_j a_j \to I_{ij}$, where $\to$ indicates the logical conditional. It is clear that definition of the possibility operator can be understood in terms of matrix multiplication in the Boolean algebra where $\vee$ is addition and $\wedge$ multiplication, so that we could write for proper finite sets of objects and attributes with cardinals $|G| = g, |M| = m$,

[2] Our names for the neighbour pair-creating functions to avoid those already taken by Formal Concept Analysis.

an incidence $I \in 2^{g \times m}$ and a set of objects $A \in 2^{g \times 1}$, $\langle I \rangle (A) = I^{\mathrm{t}} \cdot A$ with the conventional multiplication betwen boolean matrices. But that is not at all apparent in the case of the other operators.

The question arises when considering a particular incidence whether there are any more lattices related to it, and if so what their properties are. For instance, are there analogues of the constructions and methods in Formal Concept Analysis for the composition of the other operators? Are there similar operators on typical domains for data mining like $\mathbb{N}_0$ or $\mathbb{R}_0^+$? Do these operators admit a matrix representation?

In this paper we try to answer affirmatively to the questions posed above, providing analogues for the three other types of lattice stemming from incidences taking values in reflexive semifields. For that purpose apart from the wider scope of Galois connections between arbitrary orders in section 2, we reformulate in section 3 the construction of Galois connections in idempotent semimodules which are idempotent analogues of vector spaces and provide examples and a linear algebraic setting for concrete instances of these in section 4.

2 Galois Connections and Adjunctions

Let $\mathcal{P} = \langle P, \leq_{\mathcal{P}} \rangle$ and $\mathcal{Q} = \langle Q, \leq_{\mathcal{Q}} \rangle$ be partially ordered sets. We introduce the following naming conventions for the purpose of clarification [3] :

Definition 1. *1.* (λ, ρ) *is a* TYPE OO *Galois connection or* (Galois) adjunction (on the left)*, and write* $(\lambda, \rho) : \mathcal{P} \leftrightharpoons \mathcal{Q}$ *iif:* $\forall p \in P, q \in Q \quad \lambda(p) \leq_{\mathcal{Q}} q \leftrightharpoons p \leq_{\mathcal{P}} \rho(q)$ *, that is, the functions are covariant, and we say that* λ *is the* lower or left adjoint *while* ρ *is the* upper or right adjoint *.*

2. $(\rho, \lambda) : \mathcal{P} \rightleftharpoons \mathcal{Q}$ *is a* TYPE II *Galois connection or* (Galois) adjunction (on the right) *iff:* $\forall p \in P, q \in Q \quad \rho(p) \geq_{\mathcal{Q}} q \Leftrightarrow p \leq_{\mathcal{P}} \lambda(q)$ *, both functions are covariant,* ρ *is the upper adjoint, and* λ *the lower adjoint.*

3. (φ, ψ) *is a* TYPE OI *Galois connection, or* Galois Connection proper*, and write* $(\varphi, \psi) : \mathcal{P} \leftharpoonup \mathcal{Q}$ *iff:* $\forall p \in P, q \in Q \quad \varphi(p) \geq_{\mathcal{Q}} q \Leftrightarrow p \leq_{\mathcal{P}} \psi(q)$ *, that is, both functions are contravariant. For that reason they are sometimes named* contravariant or symmetric adjunctions on the right*. Note that* (ψ, φ) *is also a* TYPE OI *Galois connection.*

4. $(\triangle, \triangle')$ *is a* TYPE IO*, or* co-Galois connection*, and write* $(\triangle, \triangle') : \mathcal{P} \rightharpoonup \mathcal{Q}$ *if:* $\forall p \in P, q \in Q \quad \triangle(p) \leq_{\mathcal{Q}} q \Leftrightarrow p \geq_{\mathcal{P}} \triangle'(q)$ *, that is, both functions are contravariant. For that reason they are sometimes named* contravariant or symmetric adjunctions on the left*.* $(\triangle', \triangle)$ *is also a co-Galois connection.*

Our classification of Galois connections stresses the compositions with order- and dual order-isomorphisms, or *anti-isomorphisms*. We take the TYPE OO Galois connection to be a basic adjunction composed with an even number of

[3] For a revision of the genesis and importance of Galois Connections and adjunctions see [3], as well as a discussion of the different notation and nomenclatures for these concepts. See [4] for an early tutorial with mathematical applications in mind.

anti-isomorphism for the domain or range orders. Consequently, a TYPE II Galois connection, is a basic adjunction with an odd number of anti-isomorphisms composed on both the domain and range orders. To obtain contravariance we compose with an odd number of anti-isomorphism on the ranges to obtain a TYPE OI Galois connection. Finally, to obtain a co-Galois connection, we compose with an odd number of anti-isomorphisms only on the domain, i.e. to get a a TYPE IO Galois connection.

Table 2 summarises briefly the main properties of all types of Galois connections. Furthermore, as a sort of graphical summary, we introduce the diagram to

Table 2. Summary of Galois connections and their properties, for P, Q posets.

Left Adjunction: $(\lambda, \rho) : \mathcal{P} \leftrightarrows \mathcal{Q}$	Galois connection: $(\varphi, \psi) : \mathcal{P} \leftharpoonup\!\!\rightharpoonup \mathcal{Q}$
$\forall p \in P, q \in Q \quad \lambda(p) \leq_{\mathcal{Q}} q \leftrightarrows p \leq_{\mathcal{P}} \rho(q)$	$\forall p \in P, q \in Q \quad \varphi(p) \geq_{\mathcal{Q}} q \Leftrightarrow p \leq_{\mathcal{P}} \psi(q)$
$I_{\mathcal{P}} \leq \rho \circ \lambda$ and $I_{\mathcal{Q}} \geq \lambda \circ \rho$	$I_{\mathcal{P}} \leq \psi \circ \varphi$ and $I_{\mathcal{Q}} \leq \varphi \circ \psi$
$\lambda = \lambda \circ \rho \circ \lambda$ and $\rho = \rho \circ \lambda \circ \rho$	$\varphi = \varphi \circ \psi \circ \varphi$ and $\psi = \psi \circ \varphi \circ \psi$
λ monotone, residuated	φ antitone
ρ monotone, residual	ψ antitone
λ join-preserving, ρ meet-preserving	φ join-inverting, ψ join-inverting
co-Galois connection: $(\triangle, \triangle') : \mathcal{P} \rightharpoonup\!\!\leftharpoonup \mathcal{Q}$	**Right Adjunction: $(\rho, \lambda) : \mathcal{P} \rightleftharpoons \mathcal{Q}$**
$\forall p \in P, q \in Q \quad \triangle(p) \leq_{\mathcal{Q}} q \Leftrightarrow p \geq_{\mathcal{P}} \triangle'(q)$	$\forall p \in P, q \in Q \quad \rho(p) \geq_{\mathcal{Q}} q \Leftrightarrow p \leq_{\mathcal{P}} \lambda(q)$
$I_{\mathcal{P}} \geq \triangle' \circ \triangle$ and $I_{\mathcal{Q}} \geq \triangle \circ \triangle'$	$I_{\mathcal{P}} \geq \lambda \circ \rho$ and $I_{\mathcal{Q}} \leq \rho \circ \lambda$
$\triangle = \triangle \circ \triangle' \circ \triangle$ and $\triangle' = \triangle' \circ \triangle \circ \triangle'$	$\rho = \rho \circ \lambda \circ \rho$ and $\lambda = \lambda \circ \rho \circ \lambda$
$\triangle$ antitone	ρ monotone, residual
$\triangle$ antitone	λ monotone, residuated
$\triangle$ meet-inverting, $\triangle'$ meet-inverting	ρ meet-preserving, λ join-preserving

the upper left-hand corner of Figure 1 as the pattern that carries the structures described in ([3], §1.2) and llustrated at the top left of Figure 1:

- A closure system, $\rho(Q) = \overline{P}$, the closure range of the right adjoint (see below).
- An interior system, $\lambda(P) = \underline{Q}$, the kernel range of the left adjoint (see below).
- A closure function (also "closure operator" [6,2]) $\gamma_{\mathcal{P}} = \rho \circ \lambda \geq_{\mathcal{P}} I_{\mathcal{P}}$, from P to the closure range $\rho(Q)$, with adjoint inclusion map $\hookrightarrow_{\mathcal{P}}$, where $I_{\mathcal{P}}$ denotes the identity over P.
- A kernel function (also "interior operator" [6], "kernel operator") $\kappa_{\mathcal{P}} = \lambda \circ \rho \leq_{\mathcal{Q}} I_{\mathcal{Q}}$, from Q to the range of $\lambda(P)$, with adjoint inclusion map $\hookrightarrow_{\mathcal{Q}}$, where $I_{\mathcal{Q}}$ denotes the identity over Q.
- a *perfect adjunction* $(\tilde{\lambda}, \tilde{\rho}) : \overline{P} \leftrightarrows \underline{Q}$, that is, an order isomorphism between the closure and kernel ranges $\overline{P}$ and $\underline{Q}$.

However, a Galois connection proper can be seen in the top right of Figure 1) whose ranges are *both* closure systems and *both* compositions closure operators

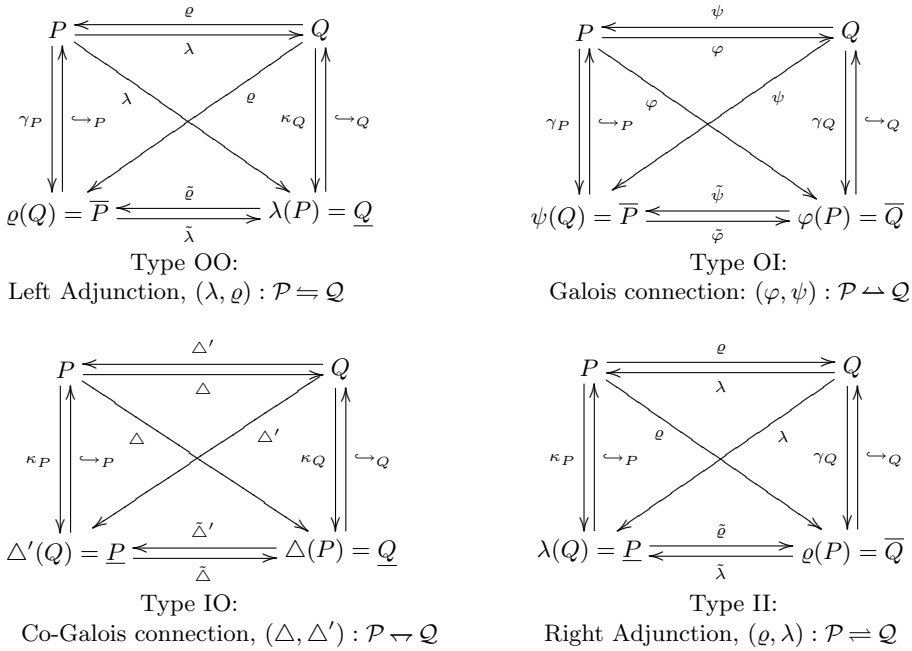

Fig. 1. Diagrams visually depicting the maps and structures involved in the adjunction on the left $(\lambda, \varrho) : P \leftrightharpoons Q$ (top left), Galois connection $(\varphi, \psi) : P \leftrightharpoons Q$ (top right), the co-Galois connection $(\triangle, \triangle') : P \rightleftharpoons Q$ (bottom left) and the adjunction on the right $(\varrho, \lambda) : P \rightleftharpoons Q$ (bottom right) between two partially ordered sets (adapted from [3,13]). Closure operators are denoted by γ_P, γ_Q , interior (kernel) operators by κ_P, κ_Q , closure systems by $\overline{P}, \overline{Q}$ and interior (kernel) systems by $\underline{P}, \underline{Q}$.

due to the dualisation of the second set (we write $\gamma_{\mathcal{Q}}$ for the new closure operator), resulting in the well-known *perfect Galois connection*, $(\tilde{\varphi}, \tilde{\psi}) : \overline{\mathcal{P}} \leftrightharpoons \overline{\mathcal{Q}}$, a pair of *dual* order isomorphism between closure ranges lying at the heart of Formal Concept Analysis. The diagrams in the bottom left and right show analogue structures for co-Galois connections and right adjunctions respectively.

As an example of all the above, consider $P = 2^G, Q = 2^M$ the powersets of a set of objects G and a set of attributes M . Then for each relation $R \in 2^{G \times M}$ we have (adapted from [4]):

- a Galois connection (TYPE OI) $(\cdot^{\,R}, \cdot^{\,R}) : 2^G \leftrightharpoons 2^M$, with dually isomorphic (closure) lattices of object and attribute sets at the heart of Formal Concept Analysis.
- a left adjunction (TYPE OO) $(\cdot\,{}^{R}_{\exists}, \cdot\,{}^{\forall}_{R}) : 2^G \leftrightharpoons 2^M$, with closure system $\left(2^M\right)^{\forall}_{R}$ that we call the *neighbourhood lattice of objects*.
- a right adjunction (TYPE II) $(\cdot\,{}^{R}_{\forall}, \cdot\,{}^{\exists}_{R}) : 2^G \rightleftharpoons 2^M$, with closure system $\left(2^G\right)^{R}_{\forall}$ that we call the *neighbourhood lattice of attributes*.

– a co-Galois connection (TYPE IO) $(\cdot\,{}^{R}_{-}, \cdot\,{}^{-}_{R}) : 2^G \rightleftarrows 2^M$, with dually isomorphic (kernel) lattices of object and attribute sets.

3 Galois Connections between Idempotent Semimodules

It is not straightforward to describe the examples in section 2 in the algebra of Boolean matrices. For this purpose, we develop the more encompassing the concept of a Galois connection between two idempotent semimodules next.

3.1 Idempotent Semirings and Semifields

Basic definitions. A *semiring* $\mathcal{S} = \langle S, \oplus, \otimes, \epsilon, e\rangle$ is a structure where the additive structure, $\langle S, \oplus, \epsilon\rangle$, is a commutative monoid and the multiplicative structure, $\langle S\backslash\{\epsilon\}, \otimes, e\rangle$, a monoid whose multiplication distributes over addition from right and left and whose neutral element is absorbing for $\otimes$, $\forall x \in K,\ \epsilon \otimes x = \epsilon$. On any semiring $\mathcal{S}$ left and right multiplications can be defined: $\mathrm{L}_a : S \to S, b \mapsto \mathrm{L}_a(b) = ab$, and $\mathrm{R}_a : S \to S, b \mapsto \mathrm{R}_a(b) = ba$. A *commutative semiring* is a one whose multiplicative structure is commutative.

For instance, let $\mathcal{S} = \langle S, \oplus, \otimes, \epsilon, e\rangle$ be a semiring. Then the semiring of (square) matrices over $\mathcal{S}$ is $\mathcal{M}_n(\mathcal{S}) = \langle S^{n\times n}, \oplus, \otimes, \mathcal{E}, E\rangle$, with $S^{n\times n}$ denoting the set of square matrices over the semiring with matrix operations: $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}, 0 \le i, j \le n$ and $(A \otimes B)_{ij} = \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj}, 0 \le i, j \le n$, null element the matrix $\mathcal{E}$, $\mathcal{E}_{ij} = \epsilon, 0 \le i, j \le n$ and unit E, $E_{ii} = e, 0 \le i \le n$, $E_{ij} = \epsilon, 0 \le i, j \le n, i \ne j$. Such semirings are not commutative in general even if $\mathcal{S}$ is, except for $\mathcal{M}_1(\mathcal{S}) = \mathcal{S}$.

A *semifield* is a semiring whose multiplicative structure $\langle S\backslash\{\epsilon\}, \otimes\rangle$ is a group, that is, there is an operation, $\cdot^{-1} : S\backslash\{\epsilon\} \to S\backslash\{\epsilon\}$ such that $\forall a \in S, a \otimes a^{-1} = a^{-1} \otimes a = e_{\mathcal{S}}$. For commutative semifields, whose multiplicative structure is a commutative group, we have $(a \otimes b)^{-1} = a^{-1} \otimes b^{-1}$.

An *idempotent semiring or dioid* (for double monoid), $\mathcal{D}$, is a semiring whose addition is idempotent, $\forall a \in D, a \oplus a = a$, that is, whose additive structure $\langle D, \oplus, \epsilon\rangle$ is an *idempotent semigroup* . Compared to a ring, an idempotent semiring crucially lacks additive inverses. All idempotent commutative monoids $\langle D, \oplus, \epsilon\rangle$ are endowed with a *natural order* $\forall a, b \in D, a \preceq b \iff a \oplus b = b$, which turns them into $\vee$-semilattices with least upper bound defined as $a \vee b = a \oplus b$. Moreover, the neutral element for the additive structure of semiring $\mathcal{D}$ is the infimum for this natural order, $\epsilon_{\mathcal{D}} = \bot_{\mathcal{D}}$. Hence all dioids are sup-semilattices $\langle D, \preceq\rangle$ with a bottom element.

A dioid whose multiplicative structure is a group is an *idempotent semifield.* The formula for the infimum in such case was already put forward by Dedekind [3]: the meet law is: $a \wedge b = a^{-1} \otimes (a \oplus b) \otimes b^{-1}$, hence idempotent semifields are already lattices. In this paper, we focus on two idempotent semifields[4]:

[4] We use := (read "becomes") to pass from abstract to concrete algebra.

1. The *maxplus semifield*, $\mathbb{R}_{\max,+} = \langle \mathbb{R} \cup \{ -\infty \}, \max, +, -\infty, 0 \rangle$ with inverse $\cdot^{-1} := -\cdot$ is an idempotent commutative semifield.
2. The *minplus semifield*, $\mathbb{R}_{\min,+} = \langle \mathbb{R} \cup \{ \infty \}, \min, +, \infty, 0 \rangle$ is an idempotent commutative semifield, with the same inverse as the previous example.

Complete Semirings and Dioids. A semiring $\mathcal{S}$ is *complete*, if for any index set I including the empty set, and any $\{a_i\}_{i \in I} \subseteq \mathcal{S}$ the (possibly infinite) summations $\bigoplus_{i \in I} a_i$ are defined and the distributivity conditions: $\left(\bigoplus_{i \in I} a_i\right) \otimes c = \bigoplus_{i \in I} (a_i \otimes c)$ and $c \otimes \left(\bigoplus_{i \in I} a_i\right) = \bigoplus_{i \in I} (c \otimes a_i)$, are satisfied. Note that for $c = e$ the above demand that infinite sums have a result. A dioid $\mathcal{D}$ is *complete*, if it is complete as a naturally ordered set $\langle D, \preceq \rangle$ and left (L_a) and right (R_a) multiplications are lower semicontinuous, that is, residuated: $\forall a, b \in D, a \preceq b, \forall c \in D, \mathrm{L}_c(a) \preceq \mathrm{L}_c(b), \mathrm{R}_c(a) \preceq \mathrm{R}_c(b)$. This implies that infinite sums are defined in terms of suprema: $\bigoplus_{i \in I} a_i = \sup_{i \in I}^{\mathcal{D}} \{a_i\}$, $\forall \{a_i\}_{i \in I} \subseteq \mathcal{D}$, with the convention that $\bigoplus_{i \in \varnothing} a_i = \epsilon$. A less strict definition is: a dioid $\mathcal{D}$ is said to be *boundedly complete*[5] if every set $\mathcal{M} \subseteq \mathcal{D}$ order-bounded from above has a least upper bound $\sup \mathcal{M} \in \mathcal{D}$.

In a complete idempotent semiring, $\mathcal{D}$, because it is a sup-semilattice $\langle D, \vee \rangle$ with bottom element ϵ, the infima of subsets also exist, hence $\mathcal{D}$ is in fact a *complete lattice* . Further, for a complete idempotent semifield, this infimum can be computed from the supremum as: $\forall a, b \in D, a \wedge b = (a^{-1} \vee b^{-1})^{-1}$ and multiplication also distributes with respect to this infimum. For instance, the Boolean semiring, $\mathcal{B} = \langle \mathbb{B}, \vee, \wedge, 0, 1 \rangle$, with $\mathbb{B} = \{0, 1\}$ is complete, idempotent and commutative.

A complete idempotent semiring $\mathcal{D}$ can never be a semifield unless it is isomorphic to the Boolean semifield $\mathcal{B}$. For instance, its maximal element $\top_{\mathcal{D}}$ satisfies $\top_{\mathcal{D}} \otimes \top_{\mathcal{D}} = \top_{\mathcal{D}}$, hence it cannot have an inverse. $\mathbb{R}_{\max,+}$ is incomplete because its bottom has no inverse in the sense that $\infty + (-\infty) = -\infty \neq 0$. For a similar reason, $\mathbb{R}_{\min,+}$ is incomplete: its bottom has no inverse: $-\infty + \infty = \infty \neq 0$. (The apparent incongruence between these sums is about to be solved.)

The Completion of Idempotent Semifields. Let a *lattice-ordered group* $\mathcal{G} = \langle G, \preceq, \otimes \rangle$ be a lattice $\langle G, \preceq \rangle$ endowed with a group operation such that the multiplications on either side are isotone (or lower semicontinuous):

$$a, b, c \in G, a \preceq b \Rightarrow c \otimes a \preceq c \otimes b, a \otimes c \preceq b \otimes c$$

A lattice-ordered group $\mathcal{G}$ is said to be *conditionally complete* if it is conditionally complete as a lattice. Every conditionally-complete lattice-ordered group is commutative in the product operation. Also, lattice-ordered semigroups which are not singletons have no least, nor greatest elements. For instance, dioids are lattice-ordered semigroups for the natural order, hence they are commutative in the product and incomplete, lacking bottom $\perp$ or top $\top$ elements, or both. However, we may complete a lattice ordered group with the *canonical enlargement construction* as follows:

[5] Also *conditionally complete* or simply *complete* in the context of dioids [5].

Construction 2 (([9,10,11,12]) Canonical enlargement of a lattice-ordered group). *For any lattice-ordered group $\mathcal{G} = \langle G, \preceq, \otimes \rangle$: adjoin two elements $\perp$ and $\top$ to G to obtain $\overline{G} = G \cup \{\perp, \top\}$ and extend the order to $\overline{G}$ as $\perp \preceq a \preceq \top, \forall a \in \bar{G}$. Then extend the product to two different operations,* upper, $\dot{\otimes}$, and lower, $\underset{\cdot}{\otimes}$, multiplications*:*

$$a \underset{\cdot}{\otimes} b = \begin{cases} \perp & \text{if } a, b \in G \cup \{\perp, \top\}, \text{with } a = \perp, \text{ or } b = \perp; \\ \top & \text{if } a, b \in G \cup \{\top\}, \text{with } a = \top, \text{ or } b = \top; \\ a \otimes b & \text{if } a, b \in G; \end{cases} \tag{1}$$

$$a \dot{\otimes} b = \begin{cases} \top & \text{if } a, b \in G \cup \{\perp, \top\}, \text{with } a = \top, \text{ or } b = \top; \\ \perp & \text{if } a, b \in G \cup \{\perp\}, \text{with } a = \perp, \text{ or } b = \perp; \\ a \otimes b & \text{if } a, b \in G; \end{cases} \tag{2}$$

to obtain the structure $\overline{\mathcal{G}} = \langle \overline{G}, \preceq, \dot{\otimes}, \underset{\cdot}{\otimes} \rangle$, known as the canonical enlargement of $\mathcal{G} = \langle G, \preceq, \otimes \rangle$ *. In this structure, $\underset{\cdot}{\otimes}$ and $\dot{\otimes}$ are associative and commutative over $\overline{G}$, as the original $\otimes$ was over G , and the isotony of the product with respect to the natural order extends to $\overline{\mathcal{G}}$. Furthermore, if e is the unit element of $\langle G, \otimes \rangle$, it is similarly the unit of $\langle \overline{G}, \dot{\otimes} \rangle$ and $\langle \overline{G}, \underset{\cdot}{\otimes} \rangle$.*

This is the basis for the completion of idempotent semifields, to follow:

Construction 3. *The* top completion *[5] of a dioid $\mathcal{D}$ is another dioid $\overline{\mathcal{D}} = \langle \overline{D}, \underset{\cdot}{\oplus}, \underset{\cdot}{\otimes}, \epsilon, e \rangle$ where: $\overline{D} = D \cup \{\top\}$ and in which $\underset{\cdot}{\otimes}$ coincides with its definition in construction 2 when $\mathcal{D}$ is considered as bearing a lattice-ordered (multiplicative semi-)group, and we extend $\oplus$ with the extra top-element:*

$$a \underset{\cdot}{\oplus} b = \begin{cases} \top & \text{if } a = \top \text{ or } b = \top; \\ a \oplus b, & \text{if } a, b \in D; \end{cases} \tag{3}$$

Construction 4 (Top Completion of an idempotent semifield). *Given an (incomplete) idempotent semifield $\mathcal{D}$, on its top enlargement as a dioid by construction 3, $\overline{\mathcal{D}}$, we extend the* notation *for the inverse with the following conventions: $\epsilon^{-1} = \top, \top^{-1} = \epsilon$. In that way we have two related complete idempotent semifield structures:*

- *a complete lattice for the natural order $\langle \overline{D}, \preceq \rangle$, the one we have been focusing on, $\overline{\mathcal{D}} = \langle \overline{D}, \underset{\cdot}{\oplus} = \vee, \underset{\cdot}{\otimes}, \perp, e \rangle$, and*
- *a complete lattice for the* dual of the natural order, $\langle \overline{D}, \preceq^d \rangle = \langle \overline{D}, \succeq \rangle$ $\overline{\mathcal{D}}^d = \langle \overline{D}, \dot{\oplus} = \wedge, \dot{\otimes}, \top, e \rangle$ *where the meet is defined (on $\mathcal{D}$) as above and the definition of $\dot{\otimes}$ follows that in construction 2 .*

Using constructions 2, 3 and 4, already invoked by Moreau [9] we have:

- The top completion of $\mathbb{R}_{\max,+}$ is $\overline{\mathbb{R}}_{\max,+} = \langle \mathbb{R} \cup \{-\infty\, \infty\}, \max, \underset{\cdot}{+}, -\infty, 0 \rangle$, the *completed Maxplus semifield.*

– The top completion of $\mathbb{R}_{\min,+}$ is $\overline{\mathbb{R}}_{\min,+} = \langle \mathbb{R} \cup \{-\infty, \infty\}, \min, \dot{+}, \infty, 0 \rangle$ the *completed Minplus semifield* .

Note that in this notation we have $-\infty + \infty = -\infty$ and $-\infty \dot{+} \infty = \infty$, which solves several issues in dealing with the separately completed dioids, as promised.

In the completed structure, we have the following De Morgan-like relations between the multiplications, their residuals and inversion:

Property 5 ([12], lemma 2.2). *In the top enlargement $\overline{\mathcal{S}}$ of any commutative semifield $\mathcal{S}$ we have:*

$$(a \underset{\cdot}{\oplus} b)^{-1} = a^{-1} \dot{\oplus} b^{-1} \qquad (a \dot{\oplus} b)^{-1} = a^{-1} \underset{\cdot}{\oplus} b^{-1} \tag{4}$$
$$(a \underset{\cdot}{\otimes} b)^{-1} = a^{-1} \dot{\otimes} b^{-1} \qquad (a \dot{\otimes} b)^{-1} = a^{-1} \underset{\cdot}{\otimes} b^{-1}$$

Furthermore if $\overline{\mathcal{S}}$ is idempotent, the residuals

$$a \underset{\cdot}{\otimes} b \preceq c \Leftrightarrow b \preceq a \underset{\cdot}{\backslash} c \Leftrightarrow a \preceq c \underset{\cdot}{/} b \qquad a \dot{\otimes} b \preceq^d c \Leftrightarrow b \preceq^d a \dot{\backslash} c \Leftrightarrow a \preceq^d c \dot{/} b \tag{5}$$

can be expressed in terms of the multiplications as:

$$a \underset{\cdot}{\backslash} c = a^{-1} \dot{\otimes} c = (a \underset{\cdot}{\otimes} c^{-1})^{-1} \qquad c \underset{\cdot}{/} a = c \dot{\otimes} a^{-1} = (c^{-1} \underset{\cdot}{\otimes} a)^{-1} \tag{6}$$
$$a \dot{\backslash} c = a^{-1} \underset{\cdot}{\otimes} c = (a \dot{\otimes} c^{-1})^{-1} \qquad c \dot{/} a = c \underset{\cdot}{\otimes} a^{-1} = (c^{-1} \dot{\otimes} a)^{-1}$$

3.2 Semimodules over Reflexive Idempotent Semifields

Basic definitions. A semimodule over a semiring is defined in a similar way to a module over a ring [1,8,7] a *left $\mathcal{S}$-semimodule*, $\mathcal{Y}$, is an additive commutative monoid $\langle Y, \oplus, \epsilon_{\mathcal{Y}} \rangle$ endowed with a map $(\lambda, y) \mapsto \lambda \otimes y$ such that $\forall \lambda, \mu \in S, \;\; y, z \in Y$, and following the convention of dropping the symbol for the scalar action and multiplication for the semiring we have:

$$(\lambda\mu)y = \lambda(\mu y) \qquad \epsilon_S y = \epsilon_{\mathcal{Y}} \tag{7}$$
$$\lambda(y \oplus z) = \lambda y \oplus \lambda z \qquad e_S y = x$$

The definition of a *right $\mathcal{S}$-semimodule*, $\mathcal{X}$, follows the same pattern with the help of a *right action*, $(\lambda, x) \mapsto x \otimes \lambda$ and similar axioms to those of (7.) A *$(\mathcal{K}, \mathcal{S})$-semimodule* is a set M endowed with left $\mathcal{K}$-semimodule and a right $\mathcal{S}$-semimodule structures, and a *$(\mathcal{K}, \mathcal{S})$-bisemimodule* a $(\mathcal{K}, \mathcal{S})$-semimodule such that the left and right multiplications commute. For a left $\mathcal{S}$-semimodule, $\mathcal{Y}$, the left and right multiplications are defined as: $\mathrm{L}^{\mathcal{S}}_{\lambda} : Y \to Y, y \mapsto \mathrm{L}^{\mathcal{S}}_{\lambda}(y) = \lambda y$, and $\mathrm{R}^{\mathcal{Y}}_{y} : S \to Y, \lambda \mapsto \mathrm{R}^{\mathcal{Y}}_{y}(\lambda) = \lambda y$. And similarly, for a right S-semimodule.

For instance, the semimodule of finite matrices $\mathcal{M}_{g \times m}(\mathcal{S}) = \langle S^{g \times m}, \oplus, \mathcal{E} \rangle$ is a $(\mathcal{M}_g(\mathcal{S}), \mathcal{M}_m(\mathcal{S}))$-bisemimodule for finite g and m, with matrix multiplication-like left and right actions and componentwise addition, and so are the set of

column vectors $\mathcal{M}_{m\times 1}(\mathcal{S})$ and the set of row vectors $\mathcal{M}_{1\times g}(\mathcal{S})$. For the completed semifields of $\overline{\mathbb{R}}_{\max,+}$ and $\overline{\mathbb{R}}_{\min,+}$, we have:

$$(A \underset{\cdot}{\otimes} B)_{ij} := \max_{k=1}^{n}(A_{ik} \underset{\cdot}{+} B_{kj}) \qquad (C \dot{\otimes} D)_{ij} := \min_{k=1}^{n}(C_{ik} \dot{+} D_{kj})$$

A left, right $\mathcal{D}$-semimodule $\mathcal{X}$ over an idempotent semiring $\mathcal{D}$ inherits the idempotent law: $\forall v \in X,\ v \oplus v = v$, which induces a *natural order* on the semimodule: $\forall v, w \in X,\ v \leq w \iff v \oplus w = w$, whereby it becomes a $\vee$-semilattice, with $\epsilon_{\mathcal{X}}$ the minimum. In the following we systematically equate idempotent $\mathcal{D}$-semimodules and semimodules over an idempotent semiring $\mathcal{D}$. When $\mathcal{D}$ is a complete idempotent semiring, a left $\mathcal{D}$-semimodule, $\mathcal{X}$ is *complete (in its natural order)* if it is complete as a naturally ordered set and its left and right multiplications are (lower semi)continuous. Trivially, it is also a complete lattice, with join and meet operations given by: $v \leq w \iff v \vee w = w \iff v \wedge w = v$. This extends naturally to right- and bisemimodules.

As in the semiring case, because of the natural order structure, the actions of idempotent semimodules admit residuation: given a complete, idempotent left $\mathcal{D}$-semimodule, $\mathcal{Y}$, we define for all $y, z \in Y$, $\lambda \in D$ the residuals $\left(\mathrm{L}_{\lambda}^{\mathcal{D}}\right)^{\#} : Y \to Y, z \mapsto \left(\mathrm{L}_{\lambda}^{\mathcal{D}}\right)^{\#}(z) = \lambda \backslash z, \left(\mathrm{R}_{y}^{\mathcal{Y}}\right)^{\#} : Y \to D, z \mapsto \left(\mathrm{R}_{y}^{\mathcal{Y}}\right)^{\#}(z) = z/y$, and likewise for a right semimodule, $\mathcal{X}$.

If $\mathcal{D}$ is idempotent (resp. complete), then finite matrix semimodules are idempotent (resp. complete) with the componentwise partial order their natural order. For $\mathcal{D}$ a completed idempotent semifield as per construction 4, the left and right residuals of $\underset{\cdot}{\otimes}$ and $\dot{\otimes}$ are:

$$(A \underset{\cdot}{\backslash} B)_{ij} = \bigoplus_{k=1}^{m}\left(A_{ki}^{-1} \dot{\otimes} B_{kj}\right) \qquad (B \underset{\cdot}{/} C)_{ij} = \bigoplus_{k=1}^{p}\left(B_{ik} \dot{\otimes} C_{jk}^{-1}\right) \tag{8}$$

$$(A \dot{\backslash} B)_{ij} = \bigoplus_{k=1}^{m}\left(A_{ki}^{-1} \underset{\cdot}{\otimes} B_{kj}\right) \qquad (B \dot{/} C)_{ij} = \bigoplus_{k=1}^{p}\left(B_{ik} \underset{\cdot}{\otimes} C_{jk}^{-1}\right)$$

with summations those of the dioid corresponding to the multiplication.

There is a remarkable operation that changes the character of a semimodule while at the same time reversing its order by means of residuation: let $\mathcal{D}$ be a complete dioid, and $\mathcal{X}$ be a complete right $\mathcal{D}$-semimodule, its *opposite semimodule* is the complete *left* $\mathcal{D}$-semimodule $\mathcal{X}^{\mathrm{op}} = \langle X, \overset{\mathrm{op}}{\oplus}, \perp_{\mathcal{X}^{\mathrm{op}}}\rangle$ with the same underlying set X, addition defined by $(x, z) \mapsto x \overset{\mathrm{op}}{\oplus} z = x \wedge z$ where the infimum is for the natural order of $\mathcal{X}$, bottom element $\perp_{\mathcal{X}^{\mathrm{op}}} = \top_{\mathcal{X}}$, and *left* action: $D \times X \to X \quad (\lambda, x) \mapsto \lambda \overset{\mathrm{op}}{\to} x = x/\lambda$. Consequently, the order of the opposite is the *dual* of the original order.

It is easy to see that $\overline{\mathbb{R}}_{\min,+}$ is precisely the complete, idempotent semiring opposite to $\overline{\mathbb{R}}_{\max,+}$, taken as a semimodule, $\overline{\mathbb{R}}_{\min,+} = (\overline{\mathbb{R}}_{\max,+})^{\mathrm{op}}$ and viceversa, $\overline{\mathbb{R}}_{\max,+} = (\overline{\mathbb{R}}_{\min,+})^{\mathrm{op}}$, since opposition of semimodules is an involution.

Finally, for an element of a semimodule over and idempotent semifield, $x \in X$, we define the *inverse* as $(x^{-1})_i = x_i^{-1}$ (which is not felicitous, given that $x^{-1} \otimes x$

is not defined in general.) However, with the precautions taken for $\overline{\mathbb{R}}_{\max,+}, \overline{\mathbb{R}}_{\min,+}$ we can write: $(x^{-1})_i := -x_i$.

Basic construction of Galois connections over reflexive semimodules. The following construction is due to Cohen et al. [1]. Let $\mathcal{D}$ be a complete dioid; for a bracket $\langle \cdot \mid \cdot \rangle : Y \times X \to Z$ between left and right $\mathcal{D}$-semimodules, $\mathcal{Y}$ and $\mathcal{X}$ respectively, onto a $\mathcal{D}$-bisemimodule $\mathcal{Z}$ and an arbitrary element $\varphi \in Z$, which we call the *pivot*, define the maps, $\cdot^*_\varphi : Y \to X$ and ${}^*_\varphi\cdot : X \to Y$:

$$y^*_\varphi = \bigvee \{ x \in X \mid \langle y \mid x \rangle \leq \varphi \} \qquad {}^*_\varphi x = \bigvee \{ y \in Y \mid \langle y \mid x \rangle \leq \varphi \} \tag{9}$$

Proposition 1 ([1], th. 42). $\left(\cdot^*_\varphi,\ {}^*_\varphi\cdot\right) : \mathcal{Y} \leftrightarrows \mathcal{X}$ *is a* TYPE OI *Galois connection.*

Note that $\mathcal{X}$ and $\mathcal{Y}$ are both already complete lattices as well as free vector spaces. However, the closure lattices $\overline{Y} = {}^*_\varphi(X)$ and $\overline{X} = (\mathcal{Y})^*_\varphi$ do not generally agree with their ambient vector spaces in their joins, but only in their meets. A *reflexive dioid*, $(\mathcal{D}, \varphi)$, is a complete dioid such that $(\langle \cdot \mid \cdot \rangle : D \times D \to D, \varphi)$ with $\langle \lambda \mid \mu \rangle = \lambda\mu$ induces a *perfect Galois connection* under construction (9), that is, a pair of mutually inverse order isomorphisms: $\forall \lambda \in D$, ${}^*_\varphi(\lambda^*_\varphi) = \lambda$, and $({}^*_\varphi\lambda)^*_\varphi = \lambda$. In reflexive dioids $\overline{X}$ and $\overline{Y}$ are actually (join-)subsemimodules of the corresponding spaces ([1], prop. 28).

This construction is affected crucially by the choice of a suitable pivot φ: if we consider the bracket to reflect a *degree of relatedness* between the elements of each pair, only those pairs $(y, x) \in Y \times X$ are considered by the connection whose degree amounts *at most* to φ . Therefore we can think of the pivot as a *maximum degree of existence* allowed for the pairs.

Finally, φ need not be unique: if $(\mathcal{D}, \varphi)$ is reflexive, for any $\lambda \in D$ invertible, $(\mathcal{D}, \varphi\lambda)$ is reflexive. Cohen et al. [1] prove that idempotent semifields are reflexive, and suggest that for the Boolean semiring we must choose $\varphi = 0_{\mathcal{B}}$, the bottom in the order. For other semifields any invertible element may be chosen, e.g. $\varphi = e_{\mathcal{D}}$.

4 Galois Connections Generated by Matrices over Completed Idempotent Semifileds

In this section we provide an easy way to build all possible Galois connections between two semimodules over an idempotent semifield. We use the Moreau notation troughout to prove that it simplifies things considerably. For all of this section, consider a completed, reflexive idempotent semiring $(\mathcal{D}, e_{\mathcal{D}})$, and let $\mathcal{Y}$ and $\mathcal{X}$ be left and right semimodules over $\mathcal{D}$ or its opposite.

Definition 6. *For* $\mathcal{Y} \cong \mathcal{D}^{1\times n}, \mathcal{X} \cong \mathcal{D}^{n\times 1}$ *and bracket* $\langle \cdot \mid \cdot \rangle_{\mathrm{OI}} : Y \times X \to D$, $\langle y \mid x \rangle_{\mathrm{OI}} = y \otimes x$ *we define a* conjugation *to be the Galois connection of type* OI *obtained from the maps in equation 9, and we write simply:* $(\cdot^*, {}^*\cdot) : \mathcal{Y} \leftrightarrows \mathcal{X}$.

Table 3. Brackets between left and right free semimodules defined over a complete idempotent semifield and its opposite.

With range in $\mathcal{D}$	With range in $\mathcal{D}^{op}$
$\langle \cdot \mid \cdot \rangle_{\mathrm{OI}} : \mathcal{D}^{1\times n} \times \mathcal{D}^{n\times 1} \to \mathcal{D}$ $\langle y \mid x \rangle_{\mathrm{OI}} = y \dot{\otimes} x$	$[\cdot \mid \cdot]_{\mathrm{OI}} : (\mathcal{D}^{op})^{1\times n} \times (\mathcal{D}^{op})^{n\times 1} \to \mathcal{D}^{op}$ $[y \mid x]_{\mathrm{OI}} = y \dot{\otimes} x$
$\langle \cdot \mid \cdot \rangle_{\mathrm{OO}} : \mathcal{D}^{1\times n} \times (\mathcal{D}^{op})^{1\times n} \to \mathcal{D}$ $\langle y \mid x \rangle_{\mathrm{OO}} = y \dot{/} x = y \dot{\otimes} x^*$	$[\cdot \mid \cdot]_{\mathrm{OO}} : (\mathcal{D}^{op})^{1\times n} \times \mathcal{D}^{1\times n} \to \mathcal{D}^{op}$ $[y \mid x]_{\mathrm{OO}} = y \dot{/} x = y \dot{\otimes} x^*$
$\langle \cdot \mid \cdot \rangle_{\mathrm{IO}} : (\mathcal{D}^{op})^{n\times 1} \times (\mathcal{D}^{op})^{1\times n} \to \mathcal{D}$ $\langle y \mid x \rangle_{\mathrm{IO}} = (x \dot{\otimes} y)^* = y^* \dot{\otimes} x^*$	$[\cdot \mid \cdot]_{\mathrm{IO}} : \mathcal{D}^{n\times 1} \times \mathcal{D}^{1\times n} \to \mathcal{D}^{op}$ $[y \mid x]_{\mathrm{IO}} = (x \dot{\otimes} y)^* = y^* \dot{\otimes} x^*$
$\langle \cdot \mid \cdot \rangle_{\mathrm{II}} : (\mathcal{D}^{op})^{n\times 1} \times \mathcal{D}^{n\times 1} \to \mathcal{D}$ $\langle y \mid x \rangle_{\mathrm{II}} = y \dot{\backslash} x = y^* \dot{\otimes} x$	$[\cdot \mid \cdot]_{\mathrm{II}} : \mathcal{D}^{n\times 1} \times (\mathcal{D}^{op})^{n\times 1} \to \mathcal{D}^{op}$ $[y \mid x]_{\mathrm{II}} = y \dot{\backslash} x = y^* \dot{\otimes} x$

By Equation (9): $y^* = y \dot{\backslash} e_{\mathcal{D}}$, ${}^*x = e_{\mathcal{D}} \dot{/} x$. For any other invertible element φ we have the *φ-conjugations*: $y^*_\varphi = y \dot{\backslash} \varphi = y \dot{\backslash} (e_{\mathcal{D}} \dot{\otimes} \varphi) = y^* \dot{\otimes} \varphi$ and ${}^*_\varphi x = \varphi \dot{\otimes} {}^*x$. Hence, the conjugations in $\overline{\mathbb{R}}_{\max,+}$ are: $y^* := -y^{\mathrm{t}}, {}^*x := -x^{\mathrm{t}}$.

Consider Table 3.We claim:

Proposition 2. *1. The brackets in the left column generate all possible types of Galois connections between $\mathcal{Y}$ and $\mathcal{X}$ by composition with adequate conjugations.*

2. The brackets in the right column generate all possible connections between the conjugates of $\mathcal{Y}$ and $\mathcal{X}$ by composition with adequate conjugations.

Proof. For 1) Bracket $\langle \cdot \mid \cdot \rangle_{\mathrm{OI}}$ generates the conjugations above, which are Galois connections of TYPE OI by Proposition 1. $\langle \cdot \mid \cdot \rangle_{\mathrm{II}}$ generates another type OI between $(\mathcal{D}^{op})^{n\times 1}$ and $\mathcal{D}^{n\times 1}$, hence pre-composing with a conjugation between $\mathcal{D}^{1\times n}$ and $(\mathcal{D}^{op})^{n\times 1}$ as defined previously obtains a right adjunction, TYPE II, between $\mathcal{D}^{1\times n}$ and $\mathcal{D}^{n\times 1}$. The procedure is exactly the same for type OO and type IO connections. For 2) the procedure is exactly the same starting from $[\cdot \mid \cdot]_{\mathrm{OI}}$ which is the one generating the Galois connection proper between $(\mathcal{D}^{op})^{1\times n}$ and $(\mathcal{D}^{op})^{n\times 1}$. □

The following proposition states, essentially, that the Galois connections over $\mathcal{D}$ and its opposite are essentially inverses (as expected from the inversion of orders between the opposite semifields):

Proposition 3. *For all brackets above, for $k \in \{\mathrm{OI},\mathrm{OO},\mathrm{IO},\mathrm{II}\}$ we have:*

$$\langle y \mid x \rangle_k = \left(\left[y^{-1} \mid x^{-1}\right]_k\right)^{-1} \qquad [y \mid x]_k = \left(\langle y^{-1} \mid x^{-1} \rangle_k\right)^{-1} \tag{10}$$

Note that such Galois connections can be built being $\mathcal{D}$ either a scalar or a matrix semiring. Hence, considering the brackets in Table 4, we claim, with a similar proof:

Table 4. Brackets between left and right free semimodules defined over a complete idempotent semifield and its opposite with the aid of matrices defined over each semifield.

With range in $\mathcal{D}$	With range in $\mathcal{D}^{op}$
$\langle \cdot \mid \cdot \rangle^R_{\mathrm{OI}} : \mathcal{D}^{1\times g} \times \mathcal{D}^{m\times 1} \to \mathcal{D}$ $\langle y \mid x \rangle^R_{\mathrm{OI}} = y \dot{\otimes} R \dot{\otimes} x$	$[\cdot \mid \cdot]^R_{\mathrm{OI}} : (\mathcal{D}^{op})^{1\times g} \times (\mathcal{D}^{op})^{m\times 1} \to \mathcal{D}^{op}$ $[y \mid x]^R_{\mathrm{OI}} = y \dot{\otimes} R \dot{\otimes} x$
$\langle \cdot \mid \cdot \rangle^R_{\mathrm{OO}} : \mathcal{D}^{1\times g} \times (\mathcal{D}^{op})^{1\times m} \to \mathcal{D}$ $\langle y \mid x \rangle^R_{\mathrm{OO}} = (y \dot{\otimes} R) \,/\, x = y \dot{\otimes} R \dot{\otimes} x^*$	$[\cdot \mid \cdot]^R_{\mathrm{OO}} : (\mathcal{D}^{op})^{1\times g} \times \mathcal{D}^{1\times m} \to \mathcal{D}^{op}$ $[y \mid x]^R_{\mathrm{OO}} = (y \dot{\otimes} R) \,/\, x = y \dot{\otimes} R \dot{\otimes} x^*$
$\langle \cdot \mid \cdot \rangle^R_{\mathrm{IO}} : (\mathcal{D}^{op})^{g\times 1} \times (\mathcal{D}^{op})^{1\times m} \to \mathcal{D}$ $\langle y \mid x \rangle^R_{\mathrm{IO}} = (x \dot{\otimes}^* R \dot{\otimes} y)^* = y^* \dot{\otimes} R \dot{\otimes} x^*$	$[\cdot \mid \cdot]^R_{\mathrm{IO}} : \mathcal{D}^{g\times 1} \times \mathcal{D}^{1\times m} \to \mathcal{D}^{op}$ $[y \mid x]^R_{\mathrm{IO}} = (x \dot{\otimes}^* R \dot{\otimes} y)^* = y^* \dot{\otimes} R \dot{\otimes} x^*$
$\langle \cdot \mid \cdot \rangle^R_{\mathrm{II}} : (\mathcal{D}^{op})^{g\times 1} \times \mathcal{D}^{m\times 1} \to \mathcal{D}$ $\langle y \mid x \rangle^R_{\mathrm{II}} = y \dot{\backslash} (R \dot{\otimes} x) = y^* \dot{\otimes} R \dot{\otimes} x$	$[\cdot \mid \cdot]^R_{\mathrm{II}} : \mathcal{D}^{g\times 1} \times (\mathcal{D}^{op})^{m\times 1} \to \mathcal{D}^{op}$ $[y \mid x]^R_{\mathrm{II}} = y \dot{\backslash} (R \dot{\otimes} x) = y^* \dot{\otimes} R \dot{\otimes} x$

Proposition 4. *1. For a given $R \in \mathcal{M}_{g\times m}(\mathcal{D})$, the brackets in the left column generate all possible types of Galois connections between the appropriate $\mathcal{Y}$ and $\mathcal{X}$ by composition with adequate conjugations.*

2. For a given $R \in \mathcal{M}_{g\times m}(\mathcal{D}^{op})$, the brackets in the right column generate all possible connections between the appropriate $\mathcal{Y}$ and $\mathcal{X}$ by composition with adequate conjugations.

Proof. The proof is straightforward following the steps of proposition 2 and a property similar to that for the brackets above. The proof for TYPE OI Galois connections can be found in [1], §4.5, as well as that of TYPE IO. □

5 Conclusion

In this paper we have provided algebraic formulae for the construction of Galois connections of all four different types viz. left and right adjunctions, Galois connections proper and co-Galois connections, between semimodules over idempotent, reflexive semifields. Although such semifields turn out to be incomplete, we have supplied a construction allowing their completion and, further, a notation, reminiscent of one introduced by Moreau, for expressing all Galois connection operators in matrix algebra.

The main scheme of combining a basic Galois connection proper plus the Cohen-Gaubert-Quadrat conjugation is already looming in [1,5]. Similarly, the use of the Moreau notation is already present in [10] in relation to co-Galois connections (TYPE IO) and the completions of certain idempotent semigroups but was not explored systematicatically there. Of course, this is exactly the way the right-axiality ($R_\forall{}^\exists$) and the co-Galois connection ($R_{\_}{}^{\_}$) where introduced in [4], but only for subsets of 2^G and 2^M, whose generalisation for other semirings is not straightforward. All in all, this shows directly that $\mathcal{K}$-Formal Concept Analysis is just one of the cases here described and indirectly the same holds for standard Formal Concept Analysis.

Acknowledgements We would like to thank M. Erné for providing initial input for Galois connection diagrams in section 2.

References

1. G. Cohen, S. Gaubert, and J.-P. Quadrat. Duality and separation theorems in idempotent semimodules. *Linear Algebra and Its Applications*, 379:395–422, 2004.
2. B. Davey and H. Priestley. *Introduction to lattices and order*. Cambridge University Press, Cambridge, UK, 2nd edition, 2002.
3. M. Erné. *Adjunctions and Galois connections: Origins, History and Development*, volume 565 of *Mathematics and Its Applications*, pages 1–138. Kluwer Academic, Dordrecht, Boston and London, 2004.
4. M. Erné, J. Koslowski, A. Melton, and G. Strecker. A primer on Galois connections. In A. Todd, editor, *Proceedings of the 1991 Summer Conference on General Topology and Applications in Honor of Mary Ellen Rudin and Her Work*, volume 704 of *Annals of the New York Academy of Sciences*, pages 103–125, Madison, WI, USA, June 1991 1993. New York Academy of Science.
5. S. Gaubert. Two lectures on max-plus algebra. Support de cours de la 26–iéme École de Printemps d'Informatique Théorique, May 1998. http://amadeus.inria.fr/gaubert/papers.html.
6. G. Gediga and I. Dütsch. Approximation operators in qualitative data analysis. Technical Report CS-03-01, Department of Computer Science, Brock University, St. Catharines, Ontario, Canada, May 2003.
7. J. S. Golan. *Power Algebras over Semirings. With Applications in Mathematics and Computer Science*, volume 488 of *Mathematics and its applications*. Kluwer Academic, Dordrecht, Boston, London, 1999.
8. J. S. Golan. *Semirings and Their Applications*. Kluwer Academic, 1999.
9. J. J. Moreau. Inf-convolution, sous-additivité, convexité des fonctions numériques. *J. Math. pures et appl.*, 49:109–154, 1970.
10. I. Singer. *Abstract Convex Analysis*. Monographs and Advanced Texts. Wiley-Interscience, 1997.
11. I. Singer. (*,s)-dualities. *Journal of Mathematical Sciences*, 115(4):2506–2541, 2003.
12. I. Singer. Some relations between linear mappings and conjugations in idempotent analysis. *Journal of Mathematical Sciences*, 115(5):2610–2630, 2003.
13. P. Taylor. *Practical Foundations of Mathematics*. Number 59 in Cambridge studies in advanced mathematics. Cambridge University Press, 1999.
14. F. J. Valverde-Albacete and C. Peláez-Moreno. Towards a generalisation of Formal Concept Analysis for data mining purposes. In R. Missaoui and J. Schmid, editors, *Concept Lattices. Proceedings of the International Conference on Formal Concept Analysis, ICFCA06, (Dresden, Germany)*, volume 3874 of *LNAI*, pages 161–176, Berlin, Heidelberg, 2006. Springer.