Finite Model Building for Propositional Gödel–Logics as an Example for Projective Logics

Markus Moschner[1]
TU Vienna

The aim of model building consists in delivering a model together with a decision result. For classical logics there are substantial achievements on automated model building [3, 4]. Although a remarkable amount of work exists for nonclassical logics (particularly for model building and automated theorem proving), the work of Negri and Plato [6] on propositional intuitionistic logic is one of the few on the problem of automatization.
My proposal refers to Gödel logics which are projective logics in the sense of [2] with respect to automated building of truth–valued models.

Prominent nonclassical logics (Gödel, intuitionistic, Łukasiewicz or lattice–valued logics) come with ordered–structures for truth–values or semantic structures (for Gödel and Łukasiewicz logics only linear truth–value structures are sufficient) Since the construction of a (counter) model is a task of semantics the usage of such orders seems suggestive. Transitivity of partial orders obviously plays a crucial role. The interpretation of the connectives refers usually to a partial ordering of its arguments and the corresponding mapping:

- conjunction — Infimum
- disjunction — Supremum
- implication — order–relation of arguments (case distinction)

Strictly speaking implication in projective logics may be defined through pure case distinction (in the finite many–valued case), thus argumentation via partial–ordering can get inelegant. Gödel logic simplifies matters through the linear order of the truth values (ensuring trichotomy between the bounds of the truth–value set) and the projector–like behavior of negation and implication([5]). Negation evaluates only to the bound values, whereas implication projects either onto the second argument or to the designated value 1. This accounts also for infinite–valued Gödel logics, since this work aims mainly at counter models the analogy to projection logics does not get senseless (on the propositional level at least).

Processing a formula from top down to its propositional variables (or constants) gives order conditions for its propositional variables:
a) for con– and disjunction there are order relations between the formula and its subformulas,
b) for implication there are order relations between its subformulas (but the value of one subformula – its succedent – may determine the value of the whole formula).
Intuitively this process is similar to a tableaux method, but "constraints" for the range of valuations are given at the nodes. These "constraints" may be order relations between different formulas — not only direct given values.
The appearance of a contradiction within these conditions means that there is no model (for the refutation — so it must be a tautology); such a contradiction has to be within every case distinction. Otherwise the conditions need not give single values for a variable, but (as a side–effect) there is information about the minimal numbers of truth–values for a nontrivial counter–model (respectively: if the conditions give 2 distinct variables which may not be interpreted to 0 or 1, a 4–valued truth–set is necessary). In general there is no restriction to finite–valued logics because only information about the structure of the (interpretations of) subformulas from finite

[1]Technische Universität Wien, Wiedner Hauptstr. 8–10, A–1040 Vienna, Austria; e–mail: *moschm@logic.at*

formulas is constructed.
My aim is an implementation that yields an automated generation of a model (if existing) for a refuted formula of propositional Gödel logics; since Gödel logics are a special case of projective logics I am interested in extending the method to classes of projective logics.

There is an elucidation of the proposed method via an extension of infinite–valued Gödel logics with 0–1–projections from [1].
The Δ–operator, representing such a 0–1–projection, gets the designated value 1 if the evaluation of the formula gets 1, otherwise 0, Such projections can express a certain order between the subformulas of an implication. Take the implicational hull with 0–1–projections of a formula (the Δ–operator gets the designated value 1 if the evaluation of the formula gets 1, otherwise 0):
$\mathcal{H}_F = \{\, \Delta(F_1 \to F_2) \,|\, F_1,\, F_2 \in Subform(F) \,\}$;
a disjunction of all conjunctions $\bigwedge_{f \in \mathcal{H}_F} (\neg^i f)$ for $i \in \{0, 1\}$ (in each conjunction every formula occurrence is either negated or unnegated) expresses all the possible schemas of valuations for a formula. The result of the proposed method (resp. one result of some case distinctions) represents a part of one of these conjunctions (there is an empty disjunction for no refutation); by laws of classical logics a completion to some of the conjunctions can be done. At the beginning of the procedure the hypothesis $\neg \Delta (1 \to A)$ is added; for a tautology always at least one conjunction admits $\Delta (1 \to A)$ for a valuation. Thus refutations of tautologies give only contradictions.

It has to be clarified if this method can be done efficiently. Further investigations have to clarify the adaptability of this method to a broader class of projective logics. But finite–valued logics can be seen as a special case of projective logics. A Within this context such a conception seems worthy of further investigations.

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