Approximate Solution of Equations with Fractional Integrals in Teaching Students

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Abstract

The scope of application of fractional calculus continues to grow today. The study of fractional integrals is included in the plans for the preparation of students in mathematical and technical areas, as well as in areas related to computer science and computer technology. When solving equations with fractional integrals, as a rule, approximate methods are often used. Computer algebra systems make it possible to perform a large number of calculations related to the solution of integro-differential problems. This article discusses the solution of a fractional integro-differential equation by the method of moments. Using Wolfram Mathematica, an approximate solution to the viscoelastic equation is found and compared with the exact solution. There are presented assignments for students that can be used in offline and online learning.

Keywords

e-learning, knowledge control, fractional integration, computer algebra system, fractional calculation, Wolfram Mathematica

1. Introduction

Students enrolled in the field "Information systems and technologies" are offered to study the course "Fractional integrals and their applications" within the disciplines of their choice. The program of this discipline includes the study of the basic properties of fractional integrals and derivatives, various types of fractional integration, applications of fractional integro-differentiation.

Fractional calculus deals with derivatives and integrals of arbitrary (real or complex) orders and has its origins in the theory of differential calculus. Most fractional differential equations, as a rule, do not have an exact analytical solution, so it is necessary to use approximate methods [1, 2].

Fractional calculus is a difficult topic for students to understand. But studying it allows one to generalize standard analysis to the case of derivatives and integrals of fractional orders; it is used to create mathematical models of the real world, where conventional analysis does not allow building adequate mathematical models. We believe that solving problems of finding an approximate solution to equations with fractional integrals will lead to an understanding of the basics of fractional calculus and the development of students' programming skills in computer mathematics systems.

Previously, we considered the quadrature formulas for the Weyl and Riemann-Liouville integrals [3, 4].

Let X and Y be arbitrary normed linear spaces, X_n and $Y_n, (n = 1,2, \dots)$, their arbitrary linear subspaces of finite dimension.

Consider the equations

$$Kx = y \ (x \in X, y \in Y), \tag{1.1}$$
$$K_n x_n = y_n \ (x_n \in X_n, y_n \in Y_n), \tag{1.2}$$

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where K and K_n are additive and homogeneous operators acting from X to Y and from X_n to Y_n, respectively.

Equation (1.2) for any fixed n is equivalent to a system of linear algebraic equations of order $N = N(n) = dimX_n$ for the expansion coefficients of the element $x_n \in X_n$ in the basis of the space X_n.

Therefore, the infinite-dimensional equation (1.1) can be replaced by the finite-dimensional equation (1.2). Lemmas giving a sufficient condition for the solvability of equation (1.1) and an estimate of the error of approximate solutions are given in [5].

2. Solution of the integro-differential equation of viscoelasticity

Consider the fractional integro-differential equation of viscoelasticity:

$$\left(D_{a+}^{\alpha} + \frac{1}{\chi_1^{\alpha}}\right)\sigma(t) = E_0\left(D_{a+}^{\alpha} + \frac{1}{\chi_2^{\alpha}}\right)e(t), \tag{2.1}$$

with the initial condition $y(0) = 0$, where $0 < \alpha < 1, t \in (a, b]$, $e(t)$ – is known, and $\sigma(t)$ – is the required function on the half-interval $[a, b]$, χ_1, χ_2 – are some constants. In this equation, the operator D^{α} is the operator of fractional differentiation and is understood in the sense of Caputo's definition.

$$D^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}dt, \qquad n-1 < \alpha \leq n, n \in \mathbb{N} \tag{2.2}$$

For the Caputo derivative

$$D^{\alpha}C = 0, (C - const), \tag{2.3}$$

$$D^{\alpha}x^{\beta} = \begin{cases} 0, \quad \beta \in \mathbb{N}_0 \text{ и } \beta < \lceil\alpha\rceil, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}, \quad \beta \in \mathbb{N}_0 \text{ и } \beta \geq \lceil\alpha\rceil \\ \beta \notin \mathbb{N} \text{ и } \beta > \lfloor\alpha\rfloor. \end{cases} \tag{2.4}$$

The Caputo fractional differentiation operator is a linear operation:

$$D^{\alpha}(\lambda f(x) + \mu g(x)) = \lambda D^{\alpha}f(x) + \mu D^{\alpha}g(x) \tag{2.5}$$

Consider the approximation of the fractional derivative using the Legendre polynomials. Legendre polynomials are defined on the interval $[-1,1]$ and can be calculated using the following recurrence formulas [6]:

$$L_{i+1}(z) = \frac{2i+1}{i+1}zL_i(z) - \frac{i}{i+1}L_{i-1}(z), i = 1,2, \ldots, \tag{2.6}$$

where $L_0(z) = 1$ and $L_1(z) = z$. In order to use these polynomials on the interval $x \in [0, 1]$, we define the so-called shifted Legendre polynomials by introducing a change in the quantity $z = 2\text{x} - 1$.

We denote the shifted Legendre polynomials $L_i(2x - 1)$ by $P_i(x)$.

Then $P_i(x)$ can be obtained as follows:

$$P_{i+1}(x) = \frac{(2i+1)(2x-1)}{(i+1)}P_i(x) - \frac{i}{i+1}P_{i-1}(x), i = 1,2, \ldots, \tag{2.7}$$

where $P_0(x) = 1$ и $P_1(x) = 2x - 1$. Analytical form of shifted Legendre polynomials

$$P_i(x) = \sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!\,x^k}{(i-k)!\,(k!)^2}. \tag{2.8}$$

Notice, that $P_i(0) = (-1)^i$ и $P_i(1) = 1$. Orthogonality condition:

$$\int_0^1 P_i(x)P_j(x)dx = \begin{cases} \dfrac{1}{2i+1} & i = j, \\ 0 & i \neq j. \end{cases} \tag{2.9}$$

The functions $y(x)$ square-integrable on $[0, 1]$ can be expressed in terms of shifted Legendre polynomials as

$$y(x) = \sum_{j=0}^{\infty} c_j P_j(x),$$

where the coefficients c_j are given by

$$c_j = (2j + 1)\int_0^1 y(x)P_j(x)dx,\ \ j = 1,2,\dots$$

In practice, only the first $(m + 1)$ items are considered as shifted Legendre polynomials.
Then we have

$$y_m(x) = \sum_{j=0}^{m} c_j P_j(x).$$

The following theorem proves the possibility of approximating the fractional derivative $y(x)$ [6].
Theorem:
Let $y(x)$ be approximated by shifted Legendre polynomials of the form

$$y_m(x) = \sum_{j=0}^{m} c_j P_j(x)$$

and suppose that $\alpha > 0$, then

$$D^{\alpha}\big(y_m(x)\big) = \sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} c_i b_{i,k}^{(\alpha)} x^{k-\alpha}, \tag{2.10}$$

where $b_{i,k}^{(\alpha)}$ is found from

$$b_{i,k}^{(\alpha)} = \frac{(-1)^{(i+k)}(i+k)!}{(i-k)!\,(k)!\,\Gamma(k+1-\alpha)} \tag{2.11}$$

Evidence: since Caputo fractional differentiation is linear operation, we have

$$D^{\alpha}\big(y_m(x)\big) = \sum_{i=0}^{m} c_i D^{\alpha}\big(P_i(x)\big). \tag{2.12}$$

Taking into account (2.3), (2.4) and (2.5) in equation (2.8), we obtain

$$D^{\alpha}P_i(x) = 0, i = 0,1,\dots,\lceil\alpha\rceil - 1, \alpha > 0. \tag{2.13}$$

Also, for $i = \lceil\alpha\rceil,\dots,m$ using (2.3), (2.4), and (2.8), we obtain

$$D^{\alpha}P_i(x) = \sum_{k=0}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!\,(k!)^2} D^{\alpha}\big(x^k\big) = \sum_{k=\lceil\alpha\rceil}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!\,(k!)\Gamma(k-\alpha+1)} x^{k-\alpha}. \tag{2.14}$$

The set of equations (2.12), (2.13), and (2.14) implies the required result.

3. Calculations in Wolfram Mathematica

In the article [7] we presented the application of the collocation method for solving equations with fractional integrals.

The equation

$$\frac{y(t)}{\chi_1^{\alpha}} + D^{\alpha}y(t) = E_0 f(t)$$

will be solved by the method of moments.

Substituting the values into this equation

$$y(t) = t^2, \chi_1^{\alpha} = \frac{1}{200}, \alpha = \frac{1}{2}, E_0 = 200\ hPa$$

(the value of the elastic modulus for iron) into this equation and simplifying it, we obtain the following:

$$t^2 + \frac{D^{\frac{1}{2}} t^2}{200} = f(t).$$

Let us find the exact value for this equation $f(t)$.

Carrying out the necessary calculations, we obtain $f(t)$:

$$\frac{8t^{3/2}}{3\sqrt{\pi}} \cdot \frac{1}{200} + t^2.$$

Next, we solve the original equation by the method of moments. We will look for an approximate value in the form of a polynomial

$$y_n(t) = \sum_{k=1}^{n} c_k \varphi_k(t).$$

The functions $\varphi_k(t)$ are of the form

$$\varphi_k(t) = \frac{k!(k+1-\alpha)t^{k-\alpha}}{\Gamma(k+2-\alpha)}.$$

Here $\varphi_k(t)$ are selected from the condition

$$D^\alpha \varphi_k(t) = t^{k-1}.$$

A proof of the form $\varphi_k(t)$ is given in [8]. Substitute the approximate value of $y_n(t)$ into the original equation:

$$\frac{y_n(t)}{\chi_1^\alpha} + D^\alpha y_n(t) = E_0 f(t),$$

$$\frac{1}{\chi_1^\alpha}\left(\sum_{k=1}^{n} c_k \varphi_k(t)\right) + D^\alpha \left(\sum_{k=1}^{n} c_k \varphi_k(t)\right) = E_0 f(t),$$

$$\left(\sum_{k=1}^{n} c_k \varphi_k(t)\right) + \frac{D^{\frac{1}{2}}}{200}\left(\sum_{k=1}^{n} c_k \varphi_k(t)\right) = f(t) \Rightarrow \sum_{k=1}^{n} c_k \varphi_k(t) + \frac{1}{200}\sum_{k=1}^{n} c_k\, D^{\frac{1}{2}} \varphi_k(t) = f(t).$$

We determine the unknown coefficients c_k from the conditions

$$\sum_{k=1}^{n} c_k \varphi_k(t) \cdot P_i + \frac{1}{200}\sum_{k=1}^{n} c_k\, D^{\frac{1}{2}} \varphi_k(t) \cdot P_i = f(t) \cdot P_i,$$

where $P_i -$ are the shifted orthogonal Legendre polynomials

$$P_i(x) = \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!\, x^k}{(i-k)!\,(k!)^2},$$

defined on the segment $[0; 1]$.

We obtain a system of linear algebraic equations of order n (with respect) related to c_k:

$$\sum_{k=1}^{n} c_k \int_0^1 \left[\varphi_k(t) - \frac{1}{200} D^{\frac{1}{2}} \varphi_k(t)\right] P_i dt = \int_0^1 f(t) P_i dt\,, \qquad i = \overline{1, n}.$$

Using the special form of the functions $\varphi_k(t)$, we obtain the property of the fractional derivative $D^\alpha \varphi_k(t) = t^{k-1}$. Thus, we have the system

$$\sum_{k=1}^{n} c_k \int_0^1 \left[\varphi_k(t) - \frac{1}{200} t^{k-1}\right] P_i(t)\, dt = \int_0^1 f(t) P_i dt\,, \qquad i = \overline{1, n}.$$

Let's write the system $n = 4$:

$$
\begin{cases}
c_1 \int_0^1 \left[\varphi_1(t) - \frac{1}{200} t^{1-1}\right] P_1 dt + c_2 \int_0^1 \left[\varphi_2(t) - \frac{1}{200} t^{2-1}\right] P_1 dt + c_3 \int_0^1 dt\, [\varphi_3(t) - \\
\qquad - \frac{1}{200} t^{3-1}\Big] P_1 + c_4 \int_0^1 \left[\varphi_4(t) - \frac{1}{200} t^{4-1}\right] P_1 dt = \int_0^1 f(t) P_1 dt\,, \\
c_1 \int_0^1 \left[\varphi_1(t) - \frac{1}{200} t^{1-1}\right] P_2 dt + c_2 \int_0^1 \left[\varphi_2(t) - \frac{1}{200} t^{2-1}\right] P_2 dt + c_3 \int_0^1 dt\, [\varphi_3(t) - \\
\qquad - \frac{1}{200} t^{3-1}\Big] P_2 + c_4 \int_0^1 \left[\varphi_4(t) - \frac{1}{200} t^{4-1}\right] P_2 dt = \int_0^1 f(t) P_2 dt\,, \\
c_1 \int_0^1 \left[\varphi_1(t) - \frac{1}{200} t^{1-1}\right] P_3 dt + c_2 \int_0^1 \left[\varphi_2(t) - \frac{1}{200} t^{2-1}\right] P_3 dt + c_3 \int_0^1 dt\, [\varphi_3(t) - \\
\qquad - \frac{1}{200} t^{3-1}\Big] P_3 + c_4 \int_0^1 \left[\varphi_4(t) - \frac{1}{200} t^{4-1}\right] P_3 dt = \int_0^1 f(t) P_3 dt\,, \\
c_1 \int_0^1 \left[\varphi_1(t) - \frac{1}{200} t^{1-1}\right] P_4 dt + c_2 \int_0^1 \left[\varphi_2(t) - \frac{1}{200} t^{2-1}\right] P_4 dt + c_3 \int_0^1 dt\, [\varphi_3(t) - \\
\qquad - \frac{1}{200} t^{3-1}\Big] P_4 + c_4 \int_0^1 \left[\varphi_4(t) - \frac{1}{200} t^{4-1}\right] P_4 dt = \int_0^1 f(t) P_4 dt\,.
\end{cases}
$$

Let's substitute the previously found values into the system:

$$f(t) = \frac{8t^{3/2}}{3\sqrt{\pi}} \cdot \frac{1}{200} + t^2$$

$$\varphi_1(t) = \frac{2t^{1/2}}{\sqrt{\pi}},\ \varphi_2(t) = \frac{8t^{3/2}}{3\sqrt{\pi}},\ \varphi_3(t) = \frac{16t^{5/2}}{5\sqrt{\pi}},\ \varphi_4(t) = \frac{128t^{7/2}}{35\sqrt{\pi}}.$$

To find the shifted Legendre polynomials in Wolfram Mathematica, we use the $LegendreP(n, 2t-1)$ library function, where n is the degree of the polynomial. The required polynomials are:

$$P_1 = 2t - 1,$$
$$P_2 = 6t^2 - 6t + 1,$$
$$P_3 = 20t^3 - 30t^2 + 12t - 1,$$
$$P_4 = 70t^4 - 140t^3 + 90t^2 - 20t + 1.$$

Received the system:

$$\begin{cases}
c_1 \int\limits_0^1 \left[\frac{2t^{1/2}}{\sqrt{\pi}} - \frac{1}{200}t^{1-1}\right] P_1 dt + c_2 \int\limits_0^1 \left[\frac{8t^{3/2}}{3\sqrt{\pi}} - \frac{1}{200}t^{2-1}\right] P_1 dt + c_3 \int\limits_0^1 dt \left[\frac{16t^{5/2}}{5\sqrt{\pi}} - \right. \\
\qquad \left. - \frac{1}{200}t^{3-1}\right] P_1 + c_4 \int\limits_0^1 \left[\frac{128t^{7/2}}{35\sqrt{\pi}} - \frac{1}{200}t^{4-1}\right] P_1 dt = \int\limits_0^1 f(t) P_1 dt\,, \\
c_1 \int\limits_0^1 \left[\frac{2t^{1/2}}{\sqrt{\pi}} - \frac{1}{200}t^{1-1}\right] P_2 dt + c_2 \int\limits_0^1 \left[\frac{8t^{3/2}}{3\sqrt{\pi}} - \frac{1}{200}t^{2-1}\right] P_2 dt + c_3 \int\limits_0^1 dt \left[\frac{16t^{5/2}}{5\sqrt{\pi}} - \right. \\
\qquad \left. - \frac{1}{200}t^{3-1}\right] P_2 + c_4 \int\limits_0^1 \left[\frac{128t^{7/2}}{35\sqrt{\pi}} - \frac{1}{200}t^{4-1}\right] P_2 dt = \int\limits_0^1 f(t) P_2 dt\,, \\
c_1 \int\limits_0^1 \left[\frac{2t^{1/2}}{\sqrt{\pi}} - \frac{1}{200}t^{1-1}\right] P_3 dt + c_2 \int\limits_0^1 \left[\frac{8t^{3/2}}{3\sqrt{\pi}} - \frac{1}{200}t^{2-1}\right] P_3 dt + c_3 \int\limits_0^1 dt \left[\frac{16t^{5/2}}{5\sqrt{\pi}} - \right. \\
\qquad \left. - \frac{1}{200}t^{3-1}\right] P_3 + c_4 \int\limits_0^1 \left[\frac{128t^{7/2}}{35\sqrt{\pi}} - \frac{1}{200}t^{4-1}\right] P_3 dt = \int\limits_0^1 f(t) P_3 dt\,, \\
c_1 \int\limits_0^1 \left[\frac{2t^{1/2}}{\sqrt{\pi}} - \frac{1}{200}t^{1-1}\right] P_4 dt + c_2 \int\limits_0^1 \left[\frac{8t^{3/2}}{3\sqrt{\pi}} - \frac{1}{200}t^{2-1}\right] P_4 dt + c_3 \int\limits_0^1 dt \left[\frac{16t^{5/2}}{5\sqrt{\pi}} - \right. \\
\qquad \left. - \frac{1}{200}t^{3-1}\right] P_4 + c_4 \int\limits_0^1 \left[\frac{128t^{7/2}}{35\sqrt{\pi}} - \frac{1}{200}t^{4-1}\right] P_4 dt = \int\limits_0^1 f(t) P_4 dt\,.
\end{cases}$$

Solving the system in Wolfram Mathematica, we get $c_1 = 0.028269$, $c_2 = 0.285231$, $c_3 = 0.417846$, $c_4 = -0.070485$.

The approximate solution has the following form, the coefficients are rounded for convenience:

$$y_4(t) = -0.031898t^{1/2} + 0.429131t^{3/2} + 0.754382t^{5/2} + -0.145433t^{7/2}.$$

The closeness of the approximate solution to the exact one by the method of moments can be estimated from Figures 1, 2 and from Table 1. The table shows the first 5 digits in the fractional part.

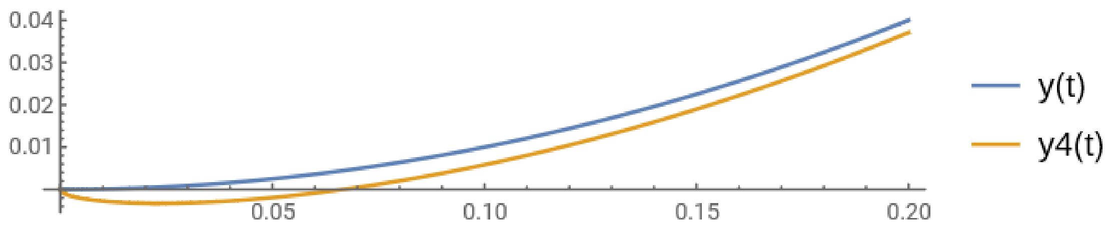

Figure 1. The graphs of the functions y (t) and $y_4(t)$ on [0; 0.2]

Table 1

Values of the functions y (t) and $y_4(t)$

	0,2	0,4	0,6	0,8	1
$y(t)$	0,04	0,16	0,36	0,64	1
$y_4(t)$	0,03709	0,15884	0,36076	0,64376	1,00618

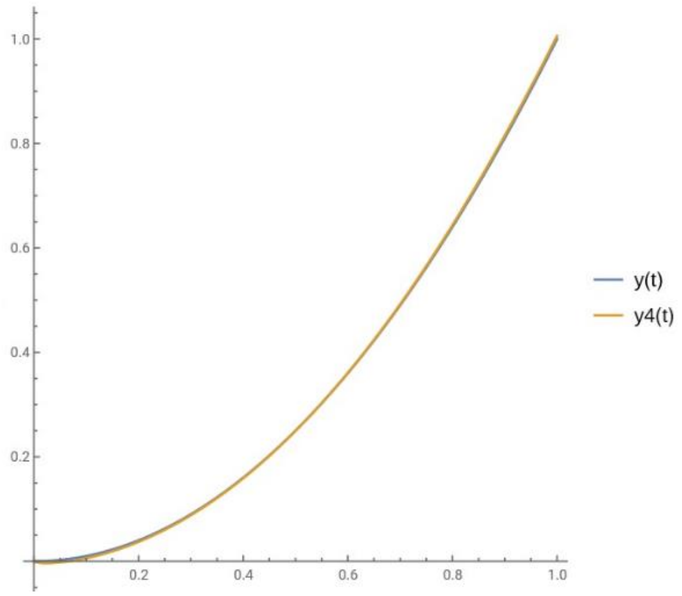

Figure 2. The graphs of the functions y (t) and $y_4(t)$ on [0; 1]

We repeated the calculations for n = 5.

An approximate solution in this case is:

$$y_5(t) = -0{,}01616\, t^{1/2} + 0{,}33758 t^{3/2} + 1{,}002305 t^{5/2} - 0.432133 t^{7/2} + 0.117484 t^{9/2}.$$

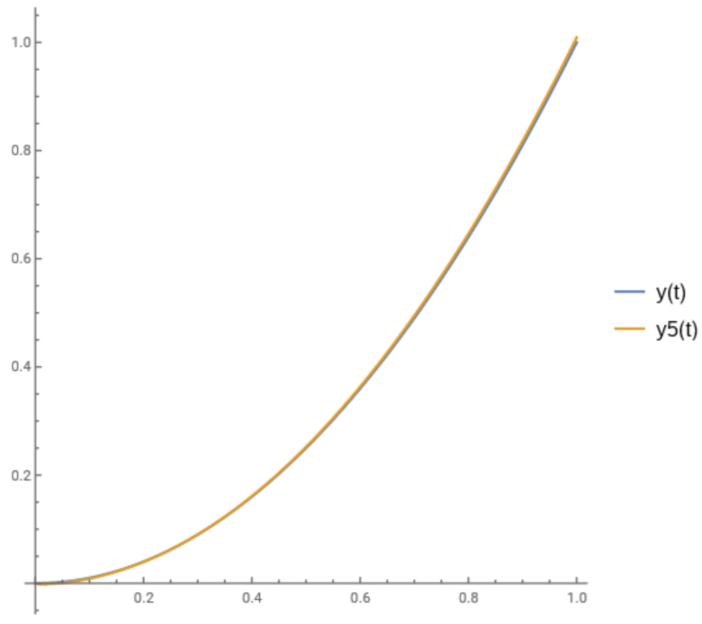

Figure 3: Graphs of functions $y(t)$ and $y_5(t)$ on [0; 1]

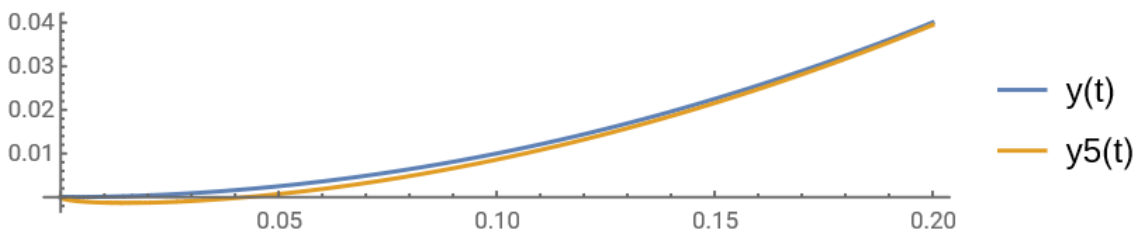

Figure 4: Graphs of functions $y(t)$ and $y_5(t)$ on [0; 0.2]

Table 2
Values of the functions y (t) and y_5(t)

	0,2	0,4	0,6	0,8	1
$y(t)$	0,04	0,16	0,36	0,64	1
$y_4(t)$	0,03943	0,16101	0,36336	0,64600	1,00907

Student assignments can be formulated as follows:

1. Write the code to find an approximate solution to the equation of viscoelasticity by the method of moments. Use Wolfram Language functions such as NIntegrate, Solve, LinearSolve.
2. Assess the accuracy of the calculations.
3. Show on the graph of the solution by means of Plot, ListPlot.

By changing n and considering the value of the elastic modulus for different substances (materials), you can get a sufficient number of tasks for a group of students. Note that students can work both in Wolfram Mathematica application and use the capabilities of the Wolfram Language Sandbox through the website https://www.wolfram.com. Such tasks can be used in traditional classroom learning and when using distance technologies.

4. Conclusions

Using Wolfram Mathematica, a numerical solution of the original integro-differential problem was made and the accuracy of calculations was estimated, and the proof of the method was provided. Solving such tasks by students will allow them to understand the basics of fractional integrodifferentiation, develop programming skills in the Wolfram Language.

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