11 Functions of More Than One Variable

11.1 Functions Given in Tabular Form

During the Winter you have probably heard weather reporters on radio or TV use the term *wind chill factor*, which is a function that depends upon wind speed and air temperature. Table 1 illustrates that wind chill (how cold it feels) in degrees Fahrenheit can be obtained as a function of wind speed in miles per hour and air temperature in degrees Fahrenheit.

Air Temperature (°F)

Wind Speed (mph)		35	30	25	20	15	10	5	0	-5	-10
	5	32	27	22	16	11	6	0	-5	-10	-15
	10	22	16	10	3	-3	-9	-15	-22	-27	-34
	15	16	9	2	-5	-11	-18	-25	-31	-38	-45
	20	12	4	-3	-10	-17	-24	-31	-39	-46	-53
	25	8	1	-7	-15	-22	-29	-36	-44	-51	-59
	30	6	-2	-10	-18	-25	-33	-41	-49	-56	-64
	35	4	-4	-12	-20	-27	-35	-43	-52	-58	-67
	40	3	-5	-13	-21	-29	-37	-45	-53	-60	-69
	45	2	-6	-14	-22	-30	-38	-46	-54	-62	-70

Table 1: Wind Chill Table

To illustrate how Table 1 is used you may ask how cold does it feel if the wind is blowing at a speed of 15 miles per hour and the air temperature is 20°F. By moving horizontally along the row headed by 15 until you reach the column headed by 20 you can see that the wind chill is -5°F.

We have seen that functions of one variable may be represented in three different ways: in table form, graphically, or by means of a formula. When data is given in tabular form one can use the Maple V procedure **surfdata** to plot a surface representing the function. This procedure is part of the Maple V library package **plots**.

```
> with(plots):
```

The procedure **surfdata** can now be invoked with the syntax:

> **surfdata**(L,<options>);

In the above the L represents a list of lists. We thus enter the data.

```
> L := [[[5,35,32],[10,35,22],[15,35,16],[20,35,12],[25,35,8],
> [30,35,6],[35,35,4],[40,35,3],[45,35,2]],[[5,30,27],[10,30,16],
> [15,30,9],[20,30,4],[25,30,1],[30,30,-2],[35,30,-4],[40,30,-5], [45,30,-
> 6]],[[5,25,22],[10,25,10], [15,25,2], [20,25,-3],[25,25,-7],[30,25,-
> 10],[35,25,-12],[40,25,-13],[45,25,-14]],[[5,20,16],[10,20,3],[15,20,-5],
> [20,20,-10],[25,20,-15],[30,20,-18],[35,20,-20],[40,20,-21],[45,20,-22]],
> [[5,15,11],[10,15,-3],[15,15,-11], [20,15,-17],[25,15,-22],[30,15,-25],
> [35,15,-27],[40,15,-29],[45,15,-30]],[[5,10,6],[10,10,-9],[15,10,-18],
> [20,10,-24],[25,10,-29],[30,10,-33],[35,10,-35],[40,10,-37],[45,10,-38]],
> [[5,5,0],[10,5,-15],[15,5,-25],[20,5,-31],[25,5,-36],[30,5,-41],[35,5,-
> 43],[40,5,-45],[45,5,-46]],[[5,0,-5],[10,0,-22],[15,0,-31],[20,0,-39],
> [25,0,-44],[30,0,-49],[35,0,-52],[40,0,-53] ,[45,0,-54]],[[5,-5,-10],[10,-
> 5,-27],[15,-5,-38],[20,-5,-46],[25,-5,-51], [30,-5,-56],[35,-5,-58],[40,-
> 5,-60], [45,-5,-62]],[[5,-10,-15],[10,-10,-34],[15,-10,-45],[20,-10,-
> 53],[25,-10,-59],[30,-10,-64],[35,-10,-67],[40,-10,-69],[45,-10,-70]]]:
```

Study the positioning of the brackets in the preceding Maple V statement, since the **surfdata** command is not very friendly. Once the data from Table 1 has been entered the surface can be plotted. See Figure 5. There are a number of options that can be used with **surfdata**, as well as any Maple V three dimensional plot routines. To get a complete list of these options issue the command:

$>$ **?plot3d[options]**

The only options that will be used in this first Maple V segment are **style**, **axes**, and **labels**.

```
> P1 := surfdata(L):
> display(P1,style=PATCHCONTOUR,axes=FRAMED,labels=[W,T,WC]);
```

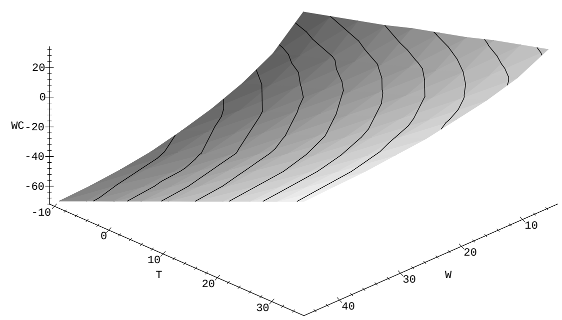

Figure 5: Wind Chill Factor Surface Plot

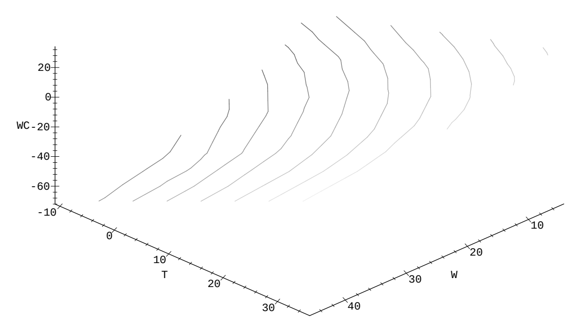

Figure 6: Contours of Wind Chill Temperatures

The option **PATCHCONTOUR** was one of seven **style** options and it caused the surface to have a shaded coloring along with contour or level curves. The option **FRAMED** we chose is one of four axes styles and causes the figure to have the axes style that it has. The option **labels** instructed Maple V to label the coordinate axes, **W** (wind speed), **T** (air temperature), **WC**, (wind chill). After you execute a Maple V three dimensional plot procedure, a new window, which contains the plot, will appear on the sceen. Once you have the plot window you may interactively change a number of the options within the plot. For example, by using the pull down menu associated with the window you may change the **style**, **color**, **axes**, or **projection** options. The **style** menu has seven different choices. Use the mouse button to select and choose the **style** option **contour**. The new plot, illustrating ten level curves of wind chill temperatures, should look like Figure 6. If you want to write a Maple V command that will give you Figure 6 directly then type in

```
> display(P1,style=CONTOUR,axes=FRAMED,labels=[W,T,WC]);
```

Figures 5 and 6 are representations on the plane of the page of a surface which lies in three space. Meteorologists use the term "extreme cold" when referring to wind chill temperatures below -20°F. Figure 7 illustrates how the wind chill surface compares with a plane parallel to the plane of wind speed and air temperature which is at a constant wind chill factor of -20°F. The following Maple V segment generates the horizontal plane $WC = -20$ and the wind chill surface on the same coordinate system.

```
> P2 := plot3d(-20,x=5..45,y=-10..35):
> display({P1,P2},style=PATCHCONTOUR,axes=FRAMED,labels=[W,T,WC]);
```

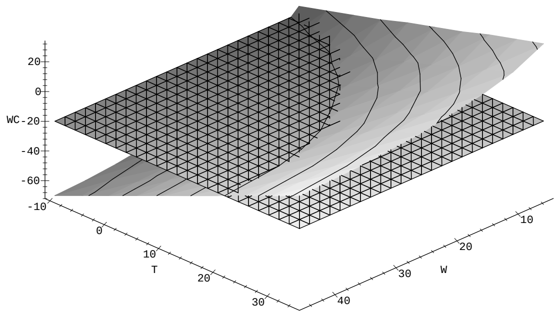

Figure 7: Extreme Cold below -20°F

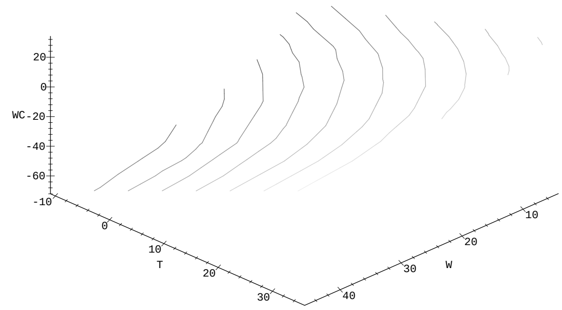

Figure 8: Contours of Wind Chill Temperatures

In Figure 7 temperatures below the plane represent wind chill temperatures that are in a state of extreme cold. Sometimes it is a help in visualizing three dimensional objects to focus on level curves or contours. If you click the mouse button when the cursor is in the window used for Figure 5, and select the **style** option **contour** then you will get a surface like the one shown in Figure 8. Type in the following Maple V segment to obtain Figure 8 directly.

```
> display(P1,style=CONTOUR,axes=FRAMED,labels=[W,T,WC]);
```

Another way of representing this function graphically is to plot the data along contour curves on planar axes. See Figure 9. This can be thought of as looking straight down the vertical axes (WC) at the surface. One way to do this is to click the mouse button in the first plot window and turn framed box until the values of **Theta** and **phi** are equal to 270 and 0 respectively. If you wish to issue a Maple V command that creates Figure 9 directly then consider the following.

```
> display({P1},style=CONTOUR,orientation = [270,0],axes=FRAMED,
> labels = [W,C,WC]);
```

One problem with Figure 8 is that that we have not indicated the values of the level curves. We can accomplish this by using the **textplot3d** command contained in the **plots** package. The syntax for this command is **textplot3d(L, options)** where L is either a list with four components or a set consisting of such lists. The first three components of the list are the coordinates in three space where you wish to have text written on your three dimensional plot and the fourth component is a string with the text that you want written. The next Maple V segment shows how to use **textplot3d** to label some of the level curves. See figure Figure 9.

```
>  T1 := textplot3d([5,30,27,‘27‘],color=BLACK):
>  T2 := textplot3d([5,20,16,‘16‘],color=BLACK):
>  T3 := textplot3d([25,30,1,‘1‘],color=BLACK):
>  T4 := textplot3d([35,20,-20,‘-20‘],color=BLACK):
>  display({P1,T1,T2,T3,T4},orientation =[270,0],style=CONTOUR,axes=FRAMED,
>  labels = [W,T,‘‘],view = [0..50,-15..40,-80..40]);
```

In the preceding command we introduced the option **view** in order to give some empty space between the graph and axes. There are other subjective terms used to describe wind chill temperature: if the wind chill is below 32°F then the weather is said to be *cold*, when below about 15°F *very cold*, when below 0°F *bitter cold*, and when below -20°F *extreme cold*. See Figure 10.

```
> T5 := textplot3d([8,35,32,‘COLD‘],color =BLACK):
```

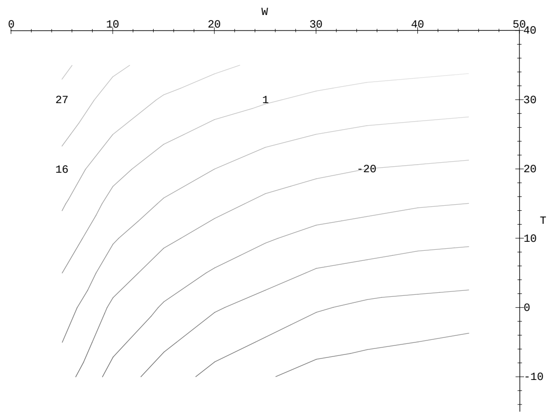

Figure 9: Labeled Contour Curves

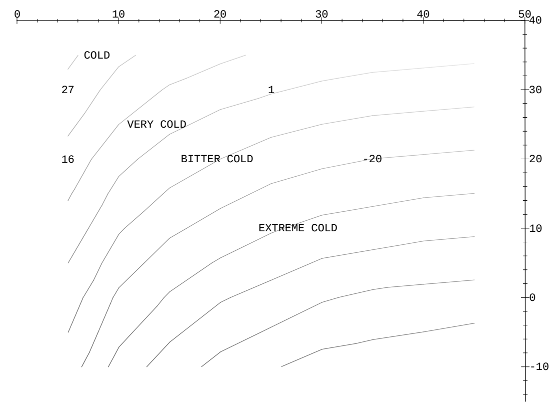

Figure 10: Contours Descriptive Text Added

```
> T6 := textplot3d([14,25,0,‘VERY COLD‘],color =BLACK):
> T7 := textplot3d([20,20,0,‘BITTER COLD‘],color =BLACK):
> T8 := textplot3d([28,10,0,‘EXTREME COLD‘],color =BLACK):
> display({P1,T1,T2,T3,T4,T5,T6,T7,T8},view = [0..50,-15..40,-80..40],
> style=contour,orientation = [270,0],axes=FRAMED);
```

In the next graph we use the plot of the plane WC = -20 to shade the region where extreme cold occurs. See Figure 11

```
> display({P1,P2,T1,T2,T3,T4},orientation=[270,0],view = [0..50,-15..40,
> -80..40],style=PATCHCONTOUR,axes=FRAMED);
```

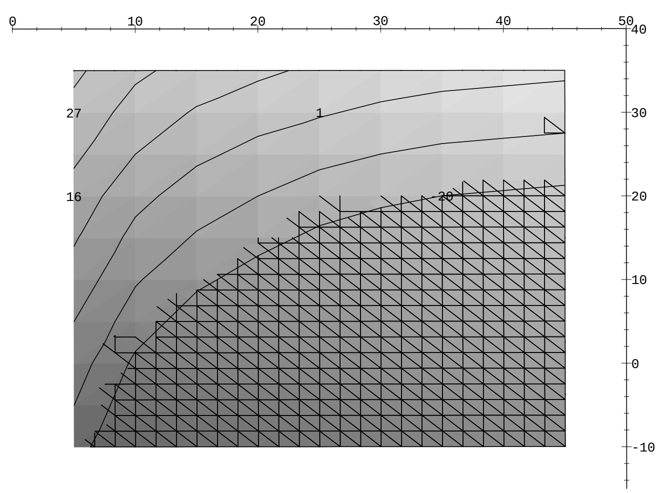

Figure 11: Contours with Region of Bitter Cold Shaded

Exercises 11.1

1. Table 2 shows the heat index as a function of temperature and humidity. The heat index is the temperature which tells you how hot it feels to you for a given air temperature and humidity.

Relative humidity (%) \ Air Temperature (°F)	70	75	80	85	90	95	100	105	110	115
0	64	69	73	78	83	87	91	95	99	103
10	65	70	75	80	85	90	95	100	105	111
20	66	72	77	82	87	93	99	105	112	120
30	67	73	78	84	90	96	104	113	123	135
40	68	74	79	86	93	101	110	123	137	151
50	69	75	81	88	96	107	120	135	150	165
60	70	76	82	90	100	114	132	149	167	187

Table 2: Heat Index

(a) If the temperature is 90°F and the humidty is 50%, how hot does it seem to be?

(b) Estimate what the relative humidity should be for 100°F to feel like 100°F.

(c) Make a lists of lists in a Maple V session and use **surfplots** to plot a graph of this data.

(d) Heat exhaustion is likely to occur when the heat index rises to 105°F. Make a Maple V plot showing contour lines of constant heat index in the relative humidity/temperature plane and shade out the region in which heat exhaustion is likely to occur.

11.2 Functions of More Than One Variable Given by Formulas

As an example of some of the things that can be done when studying functions of two variables define a function $z = f(x, y)$ as follows:

$$z = \sin(x)\cos(y).$$

As with a function of one variable you can enter it as an expression:

```
>f := sin(x)*cos(y);
```

$$f := \sin(x)\cos(y)$$

The **subs** command can be used to find the value of the expression at, say, the point $(\pi/4, \pi/3)$.

```
> subs({x=Pi/4,y=Pi/3},f);
```

$$\sin\left(\frac{1}{4}\pi\right)\cos\left(\frac{1}{3}\pi\right)$$

Perhaps this answer is better if written as follows:

```
> eval(");
```

$$\frac{1}{4}\sqrt{2}$$

It is sometimes useful to obtain the value of the function in floating point form. In this case the command **evalf** is used.

```
> evalf(");
```

$$.3535533905$$

If it is desired that the expression f be given in terms of a function, then one can define the function directly with the statement

> f := (x,y)-> sin(x)*cos(y);

or apply the command **unapply**.

```
> h := unapply(f,x,y);
```

$$h := (x, y) \rightarrow \sin(x)\cos(y)$$

In either case evaluating f at $(\pi/4, \pi/3)$ is now accomplished as follows:

```
> h(Pi/4,Pi/3);
```

$$\frac{1}{4}\sqrt{2}$$

```
> evalf(");
```

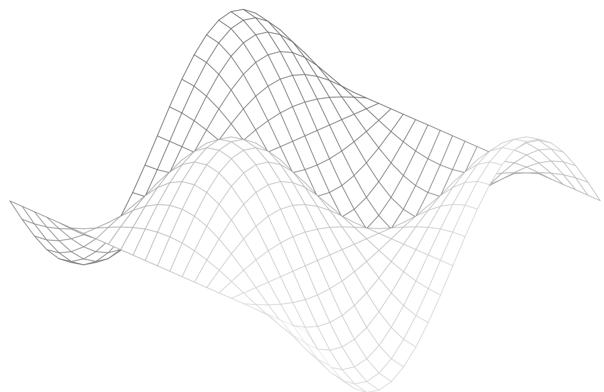

Figure 12: Plot of $z = \sin x \cos y$

.3535533905

In Maple V there are two ways to study a function: as an expression or as a Maple defined function. Which do you prefer?

The graph of a function of the form z= f(x,y) is a surface. The graph of the expression f for

$$-\pi \leq x \leq \pi, \; -\pi \leq y \leq \pi$$

is shown in Figure 12 and can be obtained with the command **plot3d** as follows:

```
> plot3d(f,x=-Pi..Pi,y=-Pi..Pi);
```

Recall that there are differences in syntax when plotting an expression, as “f” is in this example, and a function, as “h” is in this example, when using the **plot** command for the one variable case. The same plot,Figure 12, as above can be also obtained for the function h with the command **plot3d**.

```
> plot3d(h(x,y),x=-Pi..Pi,y=-Pi..Pi);
```

If the **plot3d** command is properly executed then Maple V will open a new window with the 3d plot included in it. You can view this plot from different angles as was done in the previous section. Do this by clicking on the plot with the left mouse button and holding the button down. A box will appear in place of the plot. Move the mouse while holding the left mouse button down and you view the box from different angles. The Theta and Phi in the upper left corner describe the view angles. Let go of the left mouse button. If you wish Maple to show the plot with the new angle then press and release the middle mouse button.

There are also many other options available in the 3d plot window. The **style**, **color**, **axes**, or **projection** can be changed through pull down menus. The plot can be copied and pasted into the the worksheet in the same way as was demonstrated in the introductory lesson.

HINT: You can make your 3d plot window smaller and paste the result into the worksheet to reduce the amount of paper needed when printing hard copy versions.

WARNING: It is important to keep in mind that the syntax for plotting Maple V expressions is different than the one for plotting Maple V functions. The following two Maple commands lead to an error messages and do not provide plots.

```
> plot3d(f,-Pi..Pi,-Pi..Pi); # Here f is an expression
# you should specify the ranges
   # with x= -Pi..Pi, y=-Pi..Pi
```

```
> plot3d(h(x,y),-Pi..Pi,-Pi..Pi); # Here h is a function
# you should specify the ranges
     # with x= -Pi..Pi, y=-Pi..Pi
```

Nevertheless, the following command is okay.

```
> plot3d(h,-Pi..Pi,-Pi..Pi);
```

If you wish to make a multiple plot involving two surfaces, then this can be accomplished in the same way that multiple plots can be made with curves. We now plot the function $h(x, y)$ defined above along with the expression $x^2 \sin(y)$. See Figure 13.

```
> plot3d({x^2*sin(y),h(x,y)},x=-Pi..Pi,y=-Pi..Pi);
```

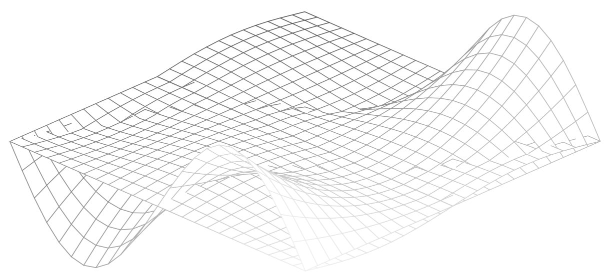

Figure 13: Plot of $z = \sin x \cos y$ and $z = x^2 \sin y$

The set of Maple V library routines that reside in **plots** contains many procedures that help in analyzing functions of two or more variables. In particular the command **contourplot** can be used to plots level curves of a function of two variables. See Figure 14.

```
> with(plots):
> contourplot(h(x,y),x=-Pi..Pi,y=-Pi..Pi);
```

You can move the plot in this window around to view this plot from different angles.

To get an idea about the relationship between **contourplot** and **plot3d** study the plot resulting from the following Maple V command.

```
> plot3d(h(x,y),x=-Pi..Pi,y=-Pi..Pi,style=contour,
             orientation = [270,0]);
```

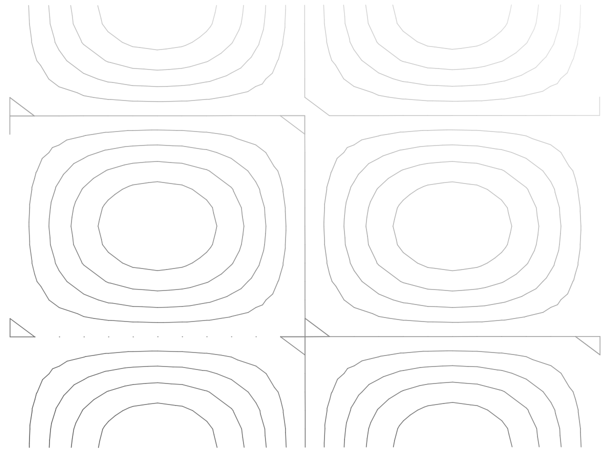

Figure 14: A **Contourplot** of $z = \sin x \cos x$

Indeed using the option **style = contour** as in the last command gives exactly the same plot, Figure 14, as **contourplot** . Try it. The above Maple V segment shows that contours are easy to obtain but the previous commands do not provide much insight into which level a particular curve corresponds.

Let us now illustrate how to use Maple V to obtain the level curve $h(x, y) = 0.6$. First we make a plot of the surface $z = h(x, y)$ and label it "P1".

```
> P1 := plot3d(h(x,y),x=-Pi..Pi,y=-Pi..Pi,orientation = [15,60]):
```

Using the command **implicitplot3d** (from **plots**) we plot the plane $h = 0.6$.

```
> P2 := implicitplot3d(z=0.6,x=-Pi..Pi,y=-Pi..Pi,z= -1..1,
                style=patchnogrid):
```

Now we display the surface $z = h(x, y)$ along with the plane $z = 0.6$. Observe that the plane intersects the surface in three curves (level curves). See Figure 15.

```
> display3d({P1,P2},tickmarks=[4,4,4],orientation = [15,60],axes=boxed);
```

You can use the **plots** Library routine **implicitplot** to obtain the level curves or contour

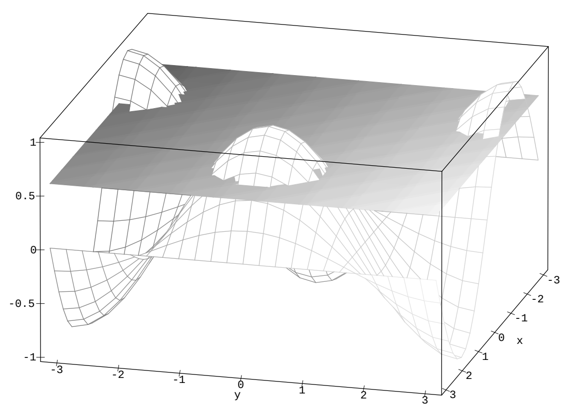

Figure 15: The Plane $z = 0.6$ and the Surface $z = \sin x \cos y$

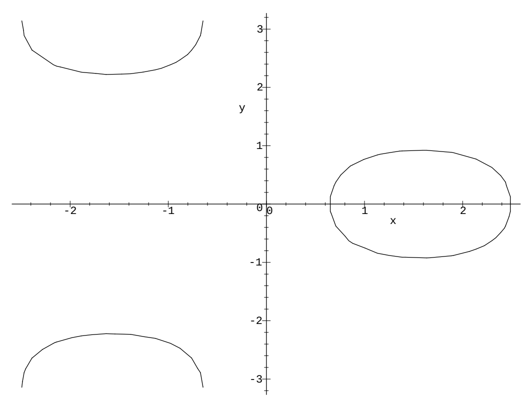

Figure 16: Contours for $\sin x \cos y = 0.6$

curves for $z = h(x, y)$ at $z = 0.6$. These curves represent the projection onto the xy-plane of the curve cut out by $z = 0.6$ from $z = h(x, y)$. See Figure 16.

```
> implicitplot(h(x,y) = 0.6, x=-Pi..Pi, y=-Pi..Pi);
```

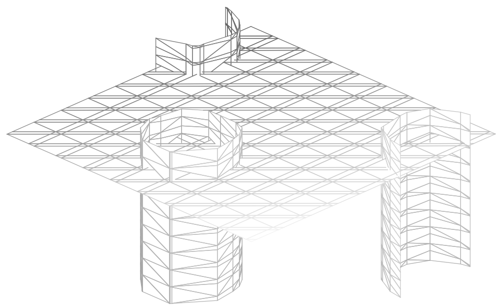

Figure 17: Level Curves for $\sin x \cos y = 0.6$

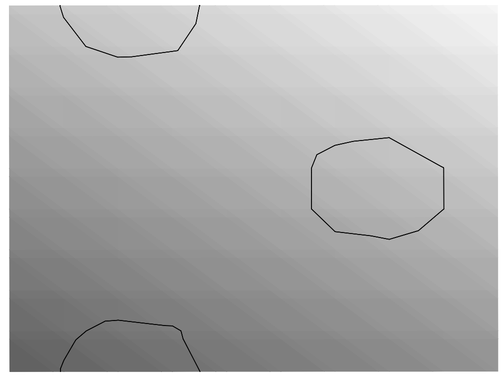

Figure 18: Contours for $\sin x \cos y = 0.6$

In the following the **plots** procedure **implicitplot3d** is used to illustrate that the level curves are also represented as the intersection of three cylinders defined implicitly in three space by

$$h(x, y) = 0.6.$$

See Figure 17.

```
> implicitplot3d({h(x,y) = 0.6, z=0.6}, x = -Pi..Pi, y=-Pi..Pi, z=-1..1);
```

Figure 18 shows the same curves after you have changed the orientation so that $\theta = 270$, and $\phi = \mathbf{0}$. The following Maple V command also creates this plot.

```
> implicitplot3d({h(x,y) = 0.6, z=0.6}, x = -Pi..Pi, y=-Pi..Pi, z=-
> 1..1,orientation=[270,0],style=PATCHCONTOUR);
```

The next example deals with a simpler function,

$$j(x, y) = x^2 + y^2,$$

that permits more explicit representations. First define j(x,y).

```
> j := (x,y) -> x^2+y^2;
```

The graph of this function over the square domain:

$$-3 \leq x \leq 3, \ -3 \leq y \leq 3$$

is shown in Figure 19.

```
> plot3d(j(x,y),x=-3..3,y=-3..3);
```

Or we can plot the graph over the over the circular domain:

$$x^2 + y^2 \leq 9.$$

See Figure 20.

```
> P1 := plot3d(j(x,y), x=-3..3, y= -sqrt(9-x^2)..sqrt(9-x^2)): ";
```

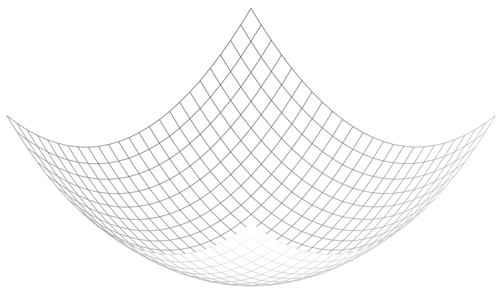

Figure 19: Plot of $z = x^2 + y^2$ over a Square Region

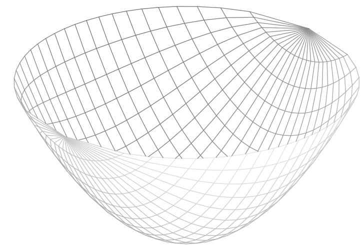

Figure 20: Plot of $z = x^2 + y^2$ over a Disk

The plane $z = 6$ is plotted next.

```
> P2 := plot3d(6,x=-3..3, y= -3..3,style = patchnogrid): ";
> plot3d(j(x,y),x=-3..3,y=-3..3);
```

The curve cut from the surface z = j(x,y) by the plane z = 6 is shown in Figure 21. The following Maple V command produces it.

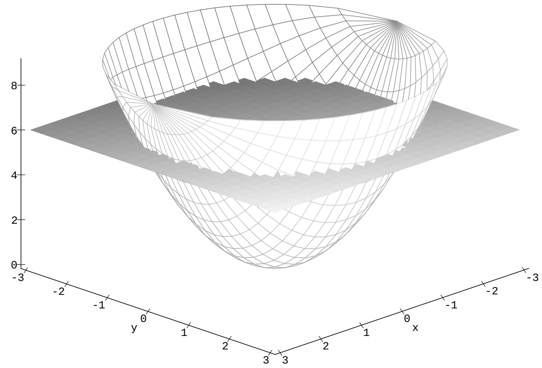

Figure 21: Plot of $z = x^2 + y^2$ and Plane $z = 6$.

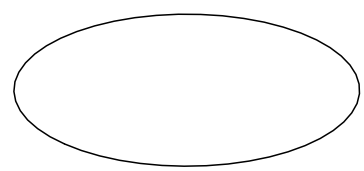

Figure 22: Curve of Intersection of $z = x^2 + y^2$ and Plane $z = 6$.

```
> display3d({P1,P2},axes=frame, tickmarks = [4,4,4],orientation=[45,60]);
```

In order to get a better idea of this intersection curve observe that any point with coordinates

$$(\sqrt{6}\cos(t), \sqrt{6}\sin(t), 6)$$

lies on both the plane $z = 6$ and the surface $z = j(x, y)$. This means that the curve of intersection of the surface and the plane is parameterized by the following three equations:

$$x = \sqrt{6}\cos(t),\ \ y = \sqrt{6}\sin(t), z = 6.$$

Using the **plots** package procedure **spacecurve** we obtain Figure 22 as follows:

```
> P3 := spacecurve([sqrt(6)*cos(t), sqrt(6)*sin(t),6],t=0..2*Pi,color=black,
> thickness=3): ";
```

Figure 23 shows curve of intersection along with the two surfaces.

```
> display3d({P1,P2,P3},tickmarks=[4,4,4],orientation = [40,120],axes=boxed);
```

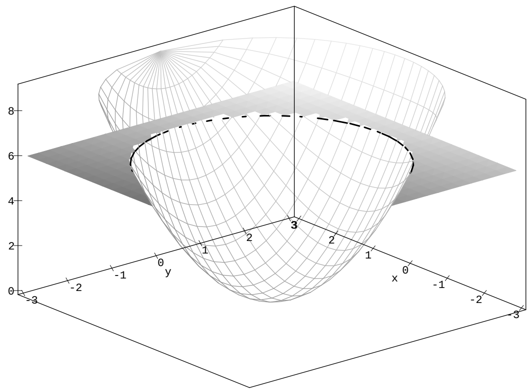

Figure 23: Plot of $z = x^2 + y^2$, $z = 6$ and the Curve of Intersection

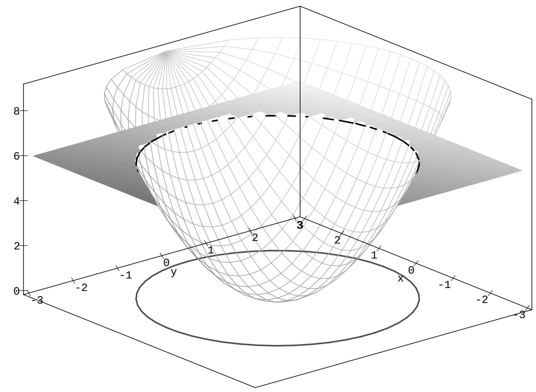

Figure 24: Curve of Intersection Projected onto xy-Plane

The contour curves are the level curves projected onto the $z = 0$ plane, and thus the following Maple V command creates a plot of this curve. See Figure 24.

```
> P4 := spacecurve([sqrt(6)*cos(t),sqrt(6)*sin(t),0],t=0..2*Pi,color=red,
> thickness=3):
```

A three dimensional plot of the curve of intersection of $z = 6$ and $z = j(x, y)$, the surfaces themselves and the contour curve on the same figure is given below.

```
> display3d({P1,P2,P3,P4},tickmarks=[4,4,4],orientation =
> [40,120],axes=boxed);
```

The contour curve is the projection of the curve of intersection of the plane $z = 6$ and the surface $z = j(x, y)$ onto the xy-plane. These two curves lie on the cylinder $j(x, y) = 6$.

```
> P5 := implicitplot3d(j(x,y)=6,x=-3..3,y=-3..3,z= 0..6): ";
```

The following plot shows all of this. See Figure 26.

```
> display3d({P1,P2,P3,P4,P5},tickmarks=[4,4,4],orientation =
>  [40,120],axes=boxed);
```

Now we illustrate how to graph certain curves that lie on the surface

$$z = x^2 + y^2.$$

First the surface is plotted using the built in Maple V command **plot3d**.

```
> P1 := plot3d(j(x,y),x=-3..3,y=-3..3): ";
```

We now wish to show the plane x = 2 cutting the surface

$$z = x^2 + y^2.$$

To do this we use the built in Maple V library routine in **plots** called **implicitplot3d** to plot the plane.

```
> P2 := implicitplot3d(x=2,x=-3..3,y=-3..3,z=0..20,style=patchnogrid):";
```

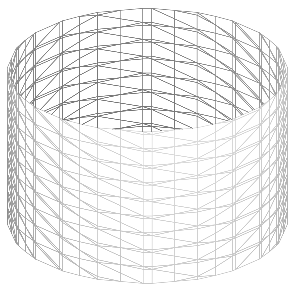

Figure 25: Cylinder $x^2 + y^2 = 6$.

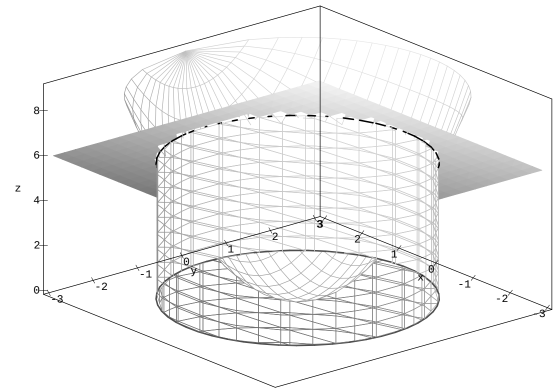

Figure 26: All of the Above

The Maple V library routine in **plots** called **display3d** is used to plot both the plane and the surface

$$z = x^2 + y^2.$$

which is shown in Figure 27.

```
> display3d({P1,P2},tickmarks=[4,4,4],orientation = [40,120], axes=framed);
```

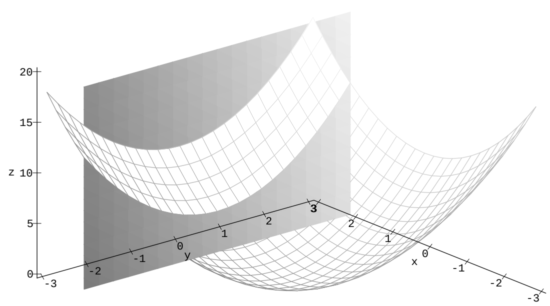

Figure 27: Intersection of Plane $x = 2$ with $z = x^2 + y^2$

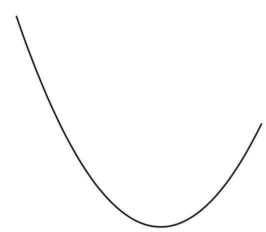

Figure 28: Curve of Intersection of Plane $x = 2$ and $z = x^2 + y^2$

The plane $x = 2$ and the surface

$$z = j(x, y) = x^2 + y^2$$

intersect in a curve. This curve is represented parametrically by the equations:

$$x = 2, \;\; y = t, \;\; z = j(2, t) = 2^2 + t^2.$$

You should make sure that you understand why all the points (x, y, z) that satisfy the above three equations also satisfy both the equation $z = j(x, y)$ and $x = 2$.

The Maple V library procedure in **plots** called **spacecurve** is now used to plot this curve. See Figure 28.

```
> P3 := spacecurve([2,t,j(2,t)],t=-3..3,thickness=3,color=black): ";
```

Now the combined plot of the surface, the plane, and the curve of intersection is obtained. See Figure 29.

```
> display3d({P1,P2,P3},tickmarks=[4,4,4],axes=framed,orientation=[40,120]);
```

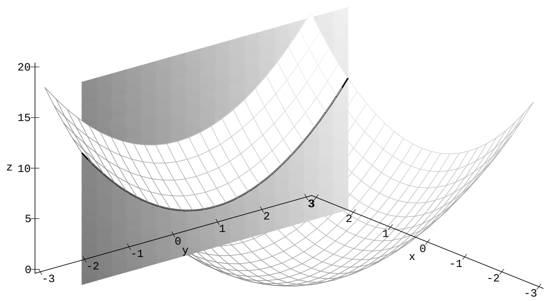

Figure 29: Intersection of Plane $x = 2,\ z = x^2 + y^2$ with Curve of Intersection

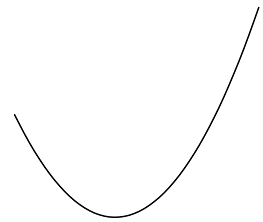

Figure 30: Curve of Intersection of Plane $y = 3$ and $z = x^2 + y^2$

Similarly other curves that lie on the intersection of the surface and a plane parallel to the xz- plane are plotted. The curve of intersection of the plane $y = 3$ and the surface $z = j(x, y)$ is represented parametrically by

$$x = t,\ y = 3,\ z = j(t, 3).$$

This curve is shown in Figure 30 and is plotted as follows:

```
> P4 := spacecurve([t,3,j(t,3)],t=-3..3,thickness=3,color=black):";
```

Finally, the curve which the plane $y = -2$ cuts out of the surface is plotted as follows:

```
> P5 := spacecurve([t,-2,j(t,-2)],t=-3..3,thickness=3,color=black):";
```

The three curves and the surface are plotted together in Figure 31 by the next Maple V command.Can you identify each of the curves in the plot with the corresponding plane?

```
> display3d({P1,P3,P4,P5},tickmarks=[4,4,4],axes=framed,style = wireframe,
> labels = [x,y,z]);
```

Exercises 11.2

1. Plot the portion of the plane
$$2x + y + 3z = 6$$
which lies in the first octant along with the line segments of intersection of the plane with the coordinate planes.

2. Plot the sphere of radius 4 centered at the origin, along with the latitudinal circle of intersection of the sphere with the plane $z = 1$.

3. Use the parametrization
$$x = (4 + 2\sin u)\cos v,\ y = (4 + 2\sin u)\sin v,\ z = 4 + 2\cos u, \quad 0 \le u \le 2\pi,\ 0 \le v \le 2\pi$$
to make a Maple V plot for the torus, created by rotating the circle of radius 2 centered at the point (4,0,0) about the z-axis.

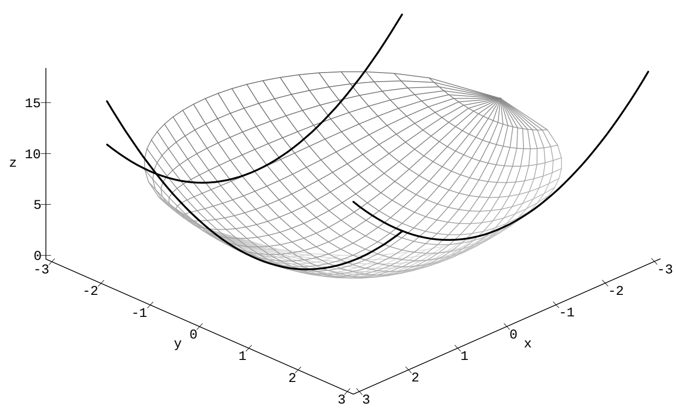

Figure 31: Plot of Three Curves and Surface $z = x^2 + y^2$

11.3 Linear Functions

Consider an equation of the form:

$$z = c + mx + ny.$$

The graph of this equation is a plane and intersects the z- axis at the point (0,0,-c). If m is not 0 the plane intersects the the x-axis at the point $(-c/m, 0, 0)$. If n is not 0 the plane intersects the y-axis at the point $(0, -c/n, 0)$. What can be said in the event that both m or n are missing ?

As an example we will now illustrate how to use Maple V to plot the plane:

$$z = 5 - 2x - y.$$

```
> P1 := plot3d(5-2*x-y,x=-5..5,y=-5..5,style=patchnogrid):";
```

In order to have a better understanding of the graph of this plane, it is sometimes helpful to plot the intersection of the plane with the coordinate planes. The intersection of the plane $x = 0$ and the plane $z = 5 - 2x - y$ has a parametric representation:

$$x = 0,\ \ y = t,\ \ z = 5 - t.$$

Show this. **Hint:** Observe that for any t the point with coordinates $(0, t, 5 - t)$ lies on both planes.

The next Maple V segment indicates how to plot a segment of this line for $0 \leq t \leq 5$ using the Maple V procedure **spacecurve** from the library package **plots**.

```
> with(plots):
> P2 := spacecurve([0,t,5-t],t=0..5,color=black):";
```

Similarly, the next two Maple commands plot the traces in the planes $y = 0$, $z = 0$ respectively.

```
> P3 := spacecurve([t,0,5-2*t],t=0..5/2,color=black):";
```

```
> P4 := spacecurve([5/2-t/2,t,0],t=0..5,color=black):";
```

Figure 32 shows a plot of all of this.

```
> display3d({P1,P2,P3,P4},tickmarks=[4,4,4],axes=normal);
```

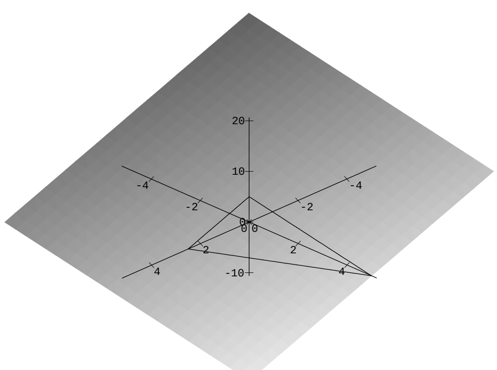

Figure 32: The planes $z = 5 - 2x - y$ and Traces with the Coordinate Planes

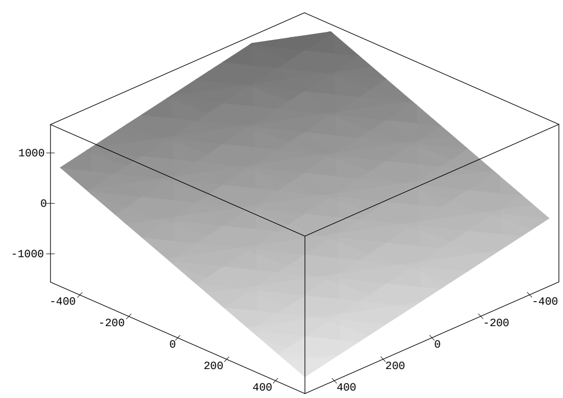

Figure 33: Plane through the Points $(100, 0, 102)$, $(101, -100, 303)$, and $(305, 0, -105)$.

Example 11.3.1 Find the equation of the plane that passes through the three points $(100, 0, 102)$, $(101, -100, 303)$, and $(305, 0, -105)$. Make a plot of this plane.

Solution The first part of this problem can be solved by substituting the coordinates of the three points into z = c + mx + ny and then solving the resulting three equations for c, m, and n using **solve**.

```
> eq1 := subs({x=100,y=0,z=102},z=c+m*x+n*y);
```

$$eq1 := 102 = c + 100m$$

```
> eq2 := subs({x=101,y=-100,z=303},z=c+m*x+n*y);
```

$$eq2 := 303 = c + 101m - 100n$$

```
> eq3 := subs({x=305,y=0,z=-105},z=c+m*x+n*y);
```

$$eq3 := -105 = c + 305m$$

```
> sol := solve({eq1,eq2,eq3},{c,m,n});
```

$$sol := \left\{ n = -\frac{10353}{5125}, m = -\frac{207}{205}, c = \frac{8322}{41} \right\}$$

Substituting this result into the general equation gives the desired equation.

```
> eqn := subs(sol,z=c+m*x+n*y);
```

$$eqn := z = \frac{8322}{41} - \frac{207\,x}{205} - \frac{10353\,y}{5125}$$

The graph of this plane can be plotted using either **plot3d** or **implicitplot3d**. See Figure 33

```
> implicitplot3d(eqn,x=-500..500,y=-500..500,z=-1500..1500,style=patchnogrid,
> axes=boxed, tickmarks = [4,4,4] );
```

WARNING: You can't plot an equation using **plot3d**.

```
> plot3d(eqn,x=-5..5,y=-5..5,style=patchnogrid,axes=boxed,
> tickmarks = [4,4,4]);
Error, (in plot3d) invalid 1st argument (the function), z = 8322/41-207/205*
x-10353/5125*y
```

But you can plot an expression. In this problem the right hand side of the equation can be plotted using the Maple command **rhs**.

```
> plot3d(rhs(eqn),x=-5..5,y=-5..5,style=patchnogrid,axes=boxed,
> tickmarks = [4,4,4]);
```

Example 11.3.2 Determine whether or not the points either of following sets of points lie on a plane.

$$\text{SetA} = \{(100, 0, 102),\ (101, -100, 303),\ (305, 0, -105),\ (1025, 364, \frac{80232617}{5125})\},$$

and

$$\text{SetB} = \{(100, 0, 102),\ (101, -100, 303),\ (305, 0, -105),\ (1025, 364, -\frac{80232617}{5125})\},$$

Solution The first three points in each set lie on the same plane as in the preceding example. Thus if the fourth point in each set satisfies the equation for the plane found in that example then it lies on the plane.

```
> subs({x=1025,y=364,z=80232617/5125},eqn);
```

$$\frac{8032617}{5125} = -\frac{8032617}{5125}$$

Thus SetA is not co-planar.

```
> subs({x=1025,y=364,z=-8032617/5125},eqn);
```

$$-\frac{8032617}{5125} = -\frac{8032617}{5125}$$

We conclude that all of the points in set SetB lie on a plane.

Exercises 11.3

1. Find the equation of the plane determined by the three points $A:\ (1, 2, -1),\ B: (2, 3, 1)$, and $C:\ (3, -1, 2)$.
2. Show that the z intercept of the plane through the points $A:\ (100, -200, 300),\ B:\ (150, 300, 200),\ C:\ (150, 300, 200)$, and $C:\ (125, 150, 100)$.

11.4 Functions of More Than Two Variables

The graph of a function of two variables is a surface or three dimensional object. The graph of a function of three variables is a three dimensional solid.

As an example consider the function

$$f(x, y, z) = x^2 + y^2 - z^2.$$

One way to obtain geometric insight into a surface is to study level curves. Analogously you can gain geometric insight into the graphs of functions of three variables by plotting level surfaces. Consider the level surfaces:

$$f(x, y, z) = 1, \;\; f(x, y, z) = -1, \;\; f(x, y, z) = 0.$$

Solving for z you can find explicit formulas, in this very special case, for each of the level surfaces:

If $f(x, y, z) = 1$ then $z = \sqrt{x^2 + y^2 - 1}$ or $z = -\sqrt{x^2 + y^2 - 1}$. If $f(x, y, z) = -1$, then $z = \sqrt{x^2 + y^2 + 1}$ or z= -$\sqrt{x^2 + y^2 + 1}$. If f(x,y,z)=0,then $z = \sqrt{x^2 + y^2}$ or $z = -\sqrt{x^2 + y^2}$.

In the first case $f(x, y, z) = 1$ we get a hyperboloid of one sheet. We now we use **implicitplot3d** which is contained in the **plots** library package. See Figure 34.

```
> with(plots):
> implicitplot3d(x^2+y^2-z^2= 1,x=-3..3,y=-3..3,z=-5..5,style=patchcontour,
> axes=framed,tickmarks=[3,3,5],orientation = [40,80]);
```

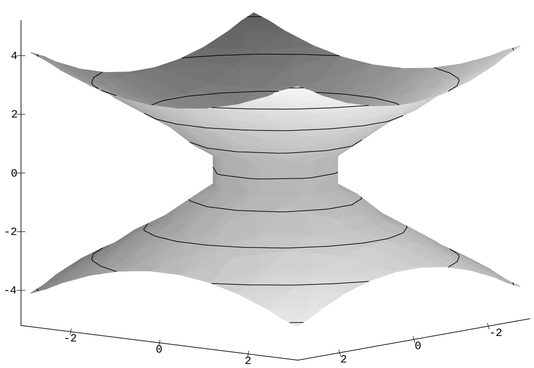

Figure 34: Level Surface $x^2 + y^2 - z^2 = 1$ Using **implicitplot3d**

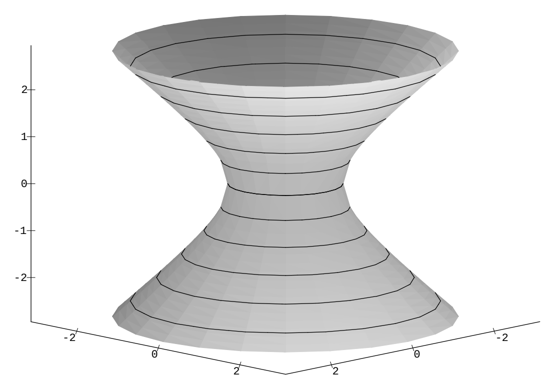

Figure 35: Level Surface $x^2 + y^2 - z^2 = 1$ Using Parameterizations

The command **implicitplot3d** gives an acceptable plot in this case, but if we parameterize the equation we can get a better plot. Observe that for any pair of numbers (r, θ) the point

$$(r\cos(\theta), r\sin(\theta), \sqrt{r^2 - 1}),$$

satisfies the equation

$$z = \sqrt{x^2 + y^2 - 1}$$

and hence lies on the hyperboloid for points in which the z-coordinate is non-negative. Similarly, the point

$$(r\cos(\theta), r\sin(\theta), -\sqrt{r^2 - 1})$$

is on the hyperboloid for z non-positive. The following plot is thus of the top half.

```
> P1 := plot3d([r*cos(theta),r*sin(theta),sqrt(r^2-1)],r=0..3,
>  theta=0..2*Pi): ";
```

Next we obtain the bottom half.

```
> P2 := plot3d([r*cos(theta),r*sin(theta),-sqrt(r^2-1)],r=0..3,
>  theta=0..2*Pi): ";
```

Both halves together. See Figure XIfig31

```
> display3d({P1,P2},style=patchcontour,orientation=[45,75],
>  axes=framed,tickmarks = [3,3,5]);
```

When $x^2+y^2-z^2=0$, the graph is a cone. The plot using **implicitplot3d** is too crude in that it does not look like a cone. See Figure 36.

```
> implicitplot3d(x^2+y^2-z^2,x=-3..3,y=-3..3,z=-5..5,style=patchcontour,
> axes=framed,tickmarks=[3,3,5],orientation = [40,80]);
```

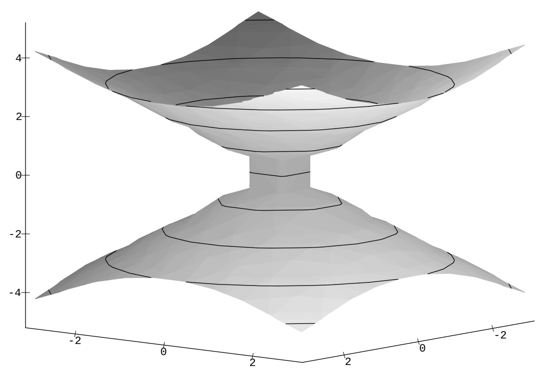

Figure 36: Level Surface $x^2+y^2-z^2=0$ Using **implicitplot3d**

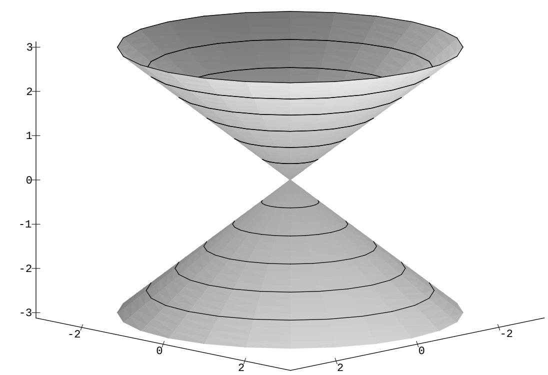

Figure 37: Level Surface $x^2+y^2-z^2=0$ Using Parameterizations

Parameterizing we get the upper and lower halves of the cone. See Figure 37.

```
> P1 := plot3d([r*cos(theta),r*sin(theta),sqrt(r^2)],r=0..3,
>  theta=0..2*Pi): ";
> P2 := plot3d([r*cos(theta),r*sin(theta),-sqrt(r^2)],r=0..3,
>  theta=0..2*Pi): ";
> display3d({P1,P2},style=patchcontour,orientation=[45,75],
>  axes=framed,tickmarks = [3,3,5]);
```

Finally, the graph of the equation $x^2+y^2-z^2=-1$ is a hyperboloid of two sheets. See Figure 38.

```
> implicitplot3d(x^2+y^2-z^2=-1,x=-3..3,y=-3..3,z=-5..5,style=patchcontour,
> axes=framed,tickmarks=[3,3,5],orientation = [40,80]);
```

Parameterizing gives a better view. See Figure 39

```
> P1 := plot3d([r*cos(theta),r*sin(theta),sqrt(r^2+1)],r=0..3,
>  theta=0..2*Pi): ";
> P2 := plot3d([r*cos(theta),r*sin(theta),-sqrt(r^2+1)],r=0..3,
>  theta=0..2*Pi): ";
> display3d({P1,P2},style=patchcontour,orientation=[45,75],
>  axes=framed,tickmarks = [3,3,5]): ";
```

Example 11.4.1 Use **implicitplot3d** to plot the Catenoid

$$\cosh z = \sqrt{x^2+y^2}.$$

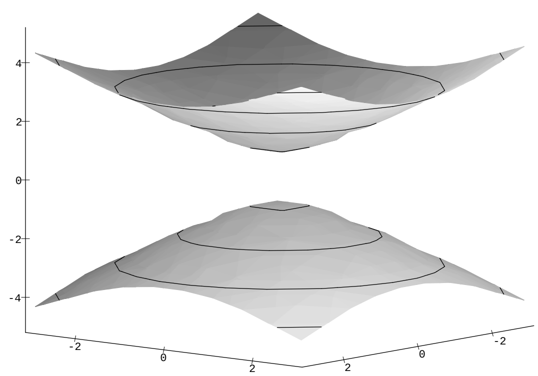

Figure 38: Level Surface $x^2 + y^2 - z^2 = -1$ Using **implicitplot3d**

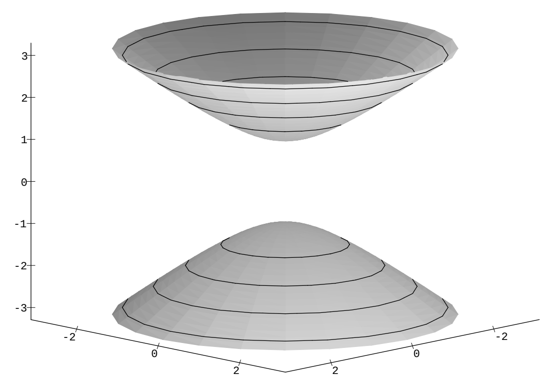

Figure 39: Level Surface $x^2 + y^2 - z^2 = -1$ Using Parameterizations

11.5 Vectors

When studying vectors using Maple V it is useful to employ procedures that are part of the Maple V Linear Algebra package called **linalg**. This package is made available by the **with(linalg)** command.

```
> with(linalg):
 Warning: new definition for   norm
Warning: new definition for   trace
```

Ignore the warnings, they will not cause you any trouble in this lesson.

Enter the following vectors

$$A =< 1, 2, 3 >, B =< 5, 0, 5 >, C =< x, y, z >$$

in order to illustrate some of the basic Maple V commands that deal with vectors. (**Note:** Throughout this book we will use the notation $< a, b, c >$ for the vector

$$a \overrightarrow{\mathbf{i}} + b \overrightarrow{\mathbf{j}} + c \overrightarrow{\mathbf{k}} .)$$

The command **vector** is available to your session once the command **with(linalg)** has been entered. There are several equivalent syntaxes for this command. We will use

$$> \textbf{vector}([x_1, \cdots, x_n]);$$

as the command to enter the vector $< x_1, \cdots, x_n >$. Now enter the vectors A, and B.

```
> A := vector([1,2,3]);
```

$$A := [\,1\,2\,3\,]$$

```
> B := vector([5,0,5]);
```

$$B := [\,5\,0\,5\,]$$

If you want to have Maple V print out the value of a vector you will need to use **evalm**, evaluate to a matrix. For example, the command

```
> A;
```

$$A$$

does not show you the values of the components of the vector A, but the commands

```
> evalm(A);
```

$$[\,1\,2\,3\,]$$

```
> print(A);
```

$$[\,1\,2\,3\,]$$

do.

The vector C has variable components and is entered in the same way:

```
> C := vector([x,y,z]);
```

$$C := [\,x\,y\,z\,]$$

Assigning values to the variables in a vector can prove a little tricky. For example.

```
> evalm(subs({x=1,y=2,z=3},C));
```

$$[\, x\; y\; z \,]$$

The following is a rather straight forward method for making a substitution into a vector.

```
> evalm([seq(subs({x=1,y=2,z=3},C[i]),i=1..3)]);
```

$$[\, 1\; 2\; 3 \,]$$

On the other hand the procedure **map** is more elegant:

```
> map(t ->subs({x=1,y=2,z=3},t),C);
```

$$[\, 1\; 2\; 3 \,]$$

In an active Maple V session you can't visually distinguish a **vector** from a **list**. To make such distinctions use the Maple V command **type**.

```
> type(",vector);
```

$$\mathrm{true}$$

The fact that the Maple V structure **vector** differs from a **list** can be illustrated as follows:

```
> type([1,2,3],vector);
```

$$\mathrm{false}$$

But, the **linalg** procedure **evalm**, evaluate matrix, converts a list to a vector.

```
> type(evalm([1,2,3]),vector);
```

$$\mathrm{true}$$

```
> type(A,vector);
```

$$\mathrm{true}$$

Observe that even though a vector and a list appear to you to be the same, Maple V regards them as different objects. A vector must be defined to the session with the command **vector**. This takes an argument which is enclosed within the "[...]".

We can add vectors. There are two ways that this can be accomplished. One way is to use the linear algebra routine **add**.

```
> S := add(A,B);
```

$$S := [\, 6\; 2\; 8 \,]$$

This looks like and is a **vector**.

```
> type(S,vector);
```

$$\mathrm{true}$$

Remember that to show the components of a vector on the screen we have to use **evalm**.

```
> evalm(S);
```

$$[6\ 2\ 8]$$

Otherwise we get the unevaluated variable:

```
> S;
```

$$S$$

Note that you can use the **add** command to add lists that look like vectors, and the result is a vector:

```
> add([1,2,3],[5,0,5]);
```

$$[6\ 2\ 8]$$

```
> type(",vector);
```

true

The following is another way to add vectors:

```
> evalm(A+B);
```

$$[6\ 2\ 8]$$

```
> type(",vector);
```

true

Also, one can add two lists and use **evalm** to convert the result to a **vector**.

```
> evalm([1,2,3]+[5,0,5]);
```

$$[6\ 2\ 8]$$

```
> type(",vector);
```

true

There are several ways to calculate the length of a vector with Maple V. Recall that if $V = < a, b, c >$ then the length of V, $||V||$, is

$$||V|| = \sqrt{a^2 + b^2 + c^2}.$$

Thus the most natural way to obtain the length of the vector A, defined above, is as follows:

```
> i := 'i'; # This resets "i" so that it can be used as an index.
```

$$i := i$$

```
> sqrt(sum(A[i]^2,i=1..3));
```

$$\sqrt{14}$$

Or if you want this result in floating point decimal to 15 digits:

```
> evalf(",15);
```

$$3.74165738677394$$

```
> sqrt(sum(C[i]^2,i=1..3));
```

$$\sqrt{x^2+y^2+z^2}$$

Within the **linalg** package is a built-in procedure **norm** which can be used to compute the length:

```
> norm(A,2);
```

$$\sqrt{14}$$

```
> norm(C,2);
```

$$\sqrt{|x|^2+|y|^2+|z|^2}$$

Make sure that you use the option **2** when using **norm** otherwise you will not get the standard length.

The vector operations dot product and cross product are available in **linalg**. Their syntax and output are obvious, if you know the mathematical meaning of the terms.

```
> dotprod(A,B);
```

$$20$$

```
> dotprod(B,A);
```

$$20$$

```
> dotprod([5,0,5],[1,2,3]);
```

$$20$$

```
> dotprod(A,C);
```

$$x+2\,y+3\,z$$

This allows us to calculate the length by yet another method.

```
> sqrt(dotprod(A,A));
```

$$\sqrt{14}$$

```
> LengthOfC := sqrt(dotprod(C,C));
```

$$LengthOfC := \sqrt{x^2+y^2+z^2}$$

The procedure **subs** can be used to assign values when the length is a variable:

```
> subs({x=3,y=4,z=5},LengthOfC);
```

$$\sqrt{50}$$

Now for some examples using **crossprod**.

```
> crossprod(A,B);
```

$$[\,10\;10\;-10\,]$$

```
> crossprod(B,A);
```

$$[\,-10\;-10\;10\,]$$

```
> crossprod([1,2,3],[5,0,5]);
```

$$[\,10\;10\;-10\,]$$

```
> crossprod(A,C);
```

$$[\,2\,z-3\,y\;3\,x-z\;y-2\,x\,]$$

One can also perform scalar multiplication using the procedure **scalarmul** from **linalg**.

```
> scalarmul(A,1/10);
```

$$\left[\frac{1}{10}\;\frac{1}{5}\;\frac{3}{10}\right]$$

```
> scalarmul([1,2,3],1/10);
```

$$\left[\frac{1}{10}\;\frac{1}{5}\;\frac{3}{10}\right]$$

```
> type(",vector);
```

$$\text{true}$$

```
> evalm(1/10*A);
```

$$\left[\frac{1}{10}\;\frac{1}{5}\;\frac{3}{10}\right]$$

```
> evalm(1/10*[1,2,3]);
```

$$\left[\frac{1}{10}\;\frac{1}{5}\;\frac{3}{10}\right]$$

One can use **scalarmul** to normalize a vector:

```
> U := scalarmul(A,1/sqrt(dotprod(A,A)));
```

$$U := \left[\frac{1}{14}\sqrt{14}\;\frac{1}{7}\sqrt{14}\;\frac{3}{14}\sqrt{14}\right]$$

```
> sqrt(dotprod(U,U));
```

$$1$$

Also the **linalg** package has its own normalization procedure **normalize**.

```
> normalize(A);
```

$$\left[\frac{1}{14}\sqrt{14}\ \frac{1}{7}\sqrt{14}\ \frac{3}{14}\sqrt{14}\right]$$

Example 11.5.1: Let $F = <-1, 1, 3>$ and $G = <2, 4, 3>$ be vectors. Find vectors $F_{parallel}$ and F_{perp} such that $F = F_{parallel} + F_{perp}$, and such that $F_{parallel}$ is parallel to G and F_{perp} is perpendicular to G.

Solution: The first step is to define the vectors F and G.

```
> F := vector([-1,1,3]); G := vector([2,4,3]);
```

$$F := [-1\,1\,3]$$

$$G := [2\,4\,3]$$

The vector $F_{parallel}$ is obtained by taking the component of F in the direction of G.

```
> Fparallel := evalm(G*dotprod(F,G)/dotprod(G,G));
```

$$Fparallel := \left[\frac{22}{29}\ \frac{44}{29}\ \frac{33}{29}\right]$$

The vector Fperp is the difference between F and $F_{parallel}$.

```
> Fperp := evalm(F-Fparallel);
```

$$Fperp := \left[\frac{-51}{29}\ \frac{-15}{29}\ \frac{54}{29}\right]$$

As a check.

```
> evalm(F) = evalm(Fparallel+Fperp);
```

$$[-1\,1\,3] = [-1\,1\,3]$$

Thus $F = F_{parallel} + F_{perp}$,

```
> dotprod(G,Fperp);
```

$$0$$

```
> crossprod(G,Fparallel);
```

$$[0\,0\,0]$$

Now do you know for sure that $F_{parallel}$ and F_{perp} are correct?

Example 11.5.2: Find the equation of the plane though the point $(1, 1, 2)$ which is perpendicular to the vector $<3, 5, 2>$.

Solution: For the point P with coordinates (x, y, z) to lie on the required plane the vector $<x-1, y-1, z-2>$ should be perpendicular to the vector $<3, 5, 2>$. Hence

```
> dotprod([3,5,2],[x-1,y-1,z-2]) = 0;
```

$$3\,x-12+5\,y+2\,z=0$$

We can show how to find the equation of a plane through three points using vector methods.

Example 11.5.3: Find the equation of the plane that passes through the three points $P=(1,0,1)$, $Q=(1,-1,3)$, and $R=(3,0,-1)$.

Solution: Since P and Q lie in the plane the vectors U and V are also in the plane where

```
> U := evalm([1,-1,3]-[1,0,1]); V := evalm([3,0,-1]-[1,0,1]);
```

$$U:=[0\ -1\ 2]$$

$$V:=[2\ 0\ -2]$$

A vector which is perpendicular to the plane is thus obtained by taking the cross product of U and V.

```
> N := crossprod(U,V);
```

$$N:=[2\ 4\ 2]$$

The equation of the plane is thus:

```
> dotprod(N,[x-1,y,z-1])=0;
```

$$2\,x-4+4\,y+2\,z=0$$

Example 11.5.4: Find the area of the triangle formed by the points with vertices:

$$A=(-122,317,615),\ \ B=(217,314,617),\ \text{and } C=(3117,-217,615).$$

Solution: The vectors AB, and AC are given by:

```
> AB := linalg[vector]([217-(-122),314-317,617-615]);
```

$$AB:=[339\ -3\ 2]$$

```
> AC := linalg[vector]([3117-(-122),-217-317,615-615]);
```

$$AC:=[3239\ -534\ 0]$$

The area of the parallelogram determined by the vectors AB and AC is equal to the length of the crossproduct

$$AB\times AC.$$

The area of the triangle is

```
> AreaTriangle := linalg[norm](linalg[crossprod](AB,AC),2)/2;
```

$$AreaTriangle := \frac{1}{2}\sqrt{29389878589}$$

Example 11.5.5: Find the volume of the parallopiped formed by the vectors

$$A =< 1, 2, 3 >, \; B =< -1, 0, 1 >, \; \text{and} C =< 2, 5, 5 > .$$

Solution:

```
> Area := abs(dotprod([1,2,3],crossprod([-1,0,1],[2,5,5])));
```

$$Area := 6$$

Exercise 11.5

1. Determine whether the plane through the three points

$$(100, -200, 300), \; (150, 300, 200), \; (125, 150, 100),$$

   is parallel to the plane through the three points

$$(121, 176, 2309), \; (250, 376, 3686), \; (300, -250, 5275).$$

   If the two planes are parallel, then find the distance between them. If the two planes are not parallel then find parametric equations for their line of intersection.

2. Find the distance between the point $(125, 265, 755)$ and the plane

$$26x - 3y - 2z + 2000 = 0.$$

3. Find the angle between the normal to the plane

$$26x - 3y - 2z + 2000 = 0$$

   and the vector

$$< 75, \; 37, \; 47 > .$$