# 14 Vector Calculus

## 14.1 Parametric Equations of Curves

In this section we will see how Maple V can be used to plot interesting curves using parametric representation.

First of all consider the circle centered at the origin with radius 2. The equation for this curve is

$$x^2 + y^2 = 4$$

If you wish plot this curve then you soon become aware that the set of points represented by the above equation cannot satisfy a relationship of the form $y = f(x)$ where $f$ is a single valued function (the vertical line test fails). The graph of the function

$$f_1(x) = \sqrt{4 - x^2}$$

is the upper semi-circle of radius 2.

```
> P1 := plot(sqrt(4-x^2),x=-2..2):";
```

Whereas the following plot gives the bottom half:

```
> P2 := plot(-sqrt(4-x^2),x=-2..2):";
```

We can now use **display** to put both plots together. See Figure 70.

```
> with(plots):
> display({P1,P2},scaling =constrained);
```

We can obtain a parameterization for the entire circle. Since the following identity is true for all real values of t

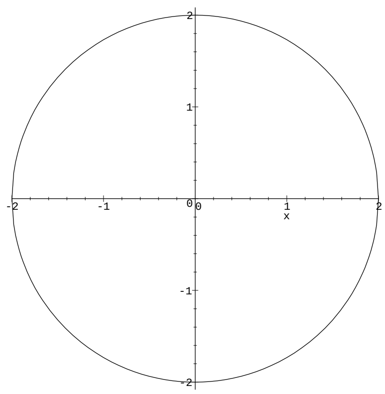


Figure 70: Circle of Radius 2

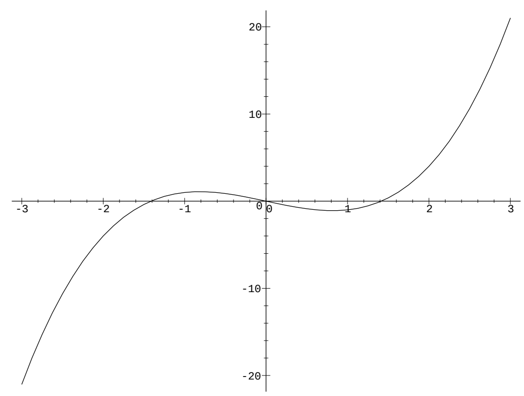


Figure 71: Parabola $x = t, \;\; y = t^3 - 2t$

$$\sin(t)^2 + \cos(t)^2 = 1$$

it follows that $x = 2\sin(t), \;\; y = 2\cos(t)$ satisfy the equation

$$x^2 + y^2 = 4$$

for all values of t. Moreover, the image of the mapping defined by the parametric equations $x = 2\cos(t), y = 2\sin(t)$ generate the entire circle, which is shown in Figure 70, as t varies for 0 to $2\pi$ with one plot statement using the **parametric** plot syntax.

```
> plot([2*cos(t),2*sin(t),t=0..2*Pi],scaling=constrained);
```

If a curve is given as an explicit function of $x$ or $y$ then it is easy to write a parameterization: if a curve is described by the relation $y = f(x)$, for $a \leq x \leq b$, then a parameterization for this curve is $x = t,\ y = f(t)$, for $a \leq t \leq b$, and, similarly, if a curveis described as $x = g(y)$, then the parameterization can be of the form $x = g(t),\ y = t$. For example, suppose

$$y = x^3 - 2\,x$$

is given. Then either of the following plot commands will give the Figure 71. for any range, say, $-3 \leq x \leq 3$.

```
> plot(x^3-2*x,x=-3..3);
> plot([t,t^3-2*t,t=-3..3]);
```

Similarly if $x = y^2 - y$ is the function you wish to plot then the following command does the trick. See Figure 72.

```
> plot([t^2-t,t,t=-2..2]);
```

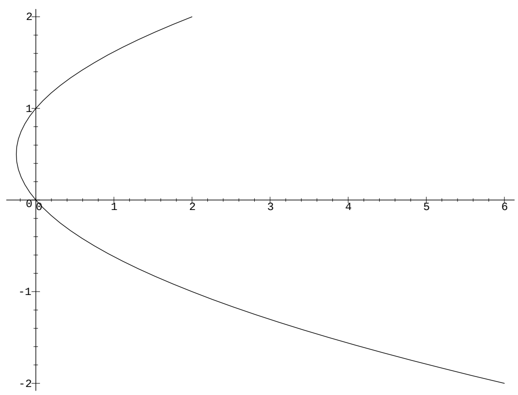

Figure 72: Parabola $x = t^2 - t, y = t$

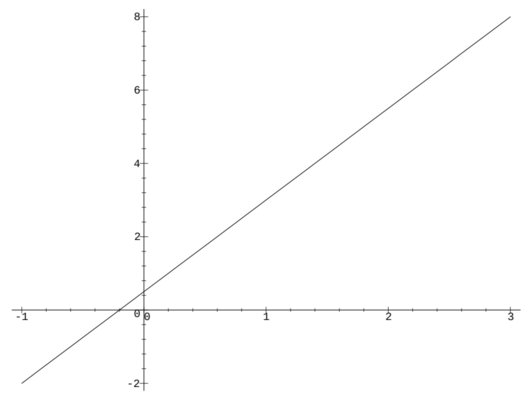

Figure 73: Line $x = 1 + 2t,\ y = 3 + 5t$

Straight lines are very easy to parameterize. For example if $y = mx + b$ is the equation of the line then $x = t,\ y = mt + b$ gives a parameterization. But you can sometimes write the parametric equations directly from their geometric description. For example, a portion of the straight line through the point (1,3) parallel to the vector $< 2, 5 >$ has parametric equations $x = 1 + 2t,\ y = 3 + 5t$, is shown in Figure 73 and can be plotted by the command:

```
> plot([1+2*t,3+5*t,t=-1..1]);
```

On occasion it possible to plot curves that would be very complicated to express in even implicit algebraic terms. For example, the following curve is known as a Lissajous Figure and is shown in Figure 74.

```
> plot([cos(3*t),sin(5*t),t=0..2*Pi],scaling=constrained);
```

Solutions of differential equations are usually represented in parametric form. For example $x = \cos(2t),\ y = -2\sin(2t)$ is a solution of the initial value problem:

$$\begin{aligned} \frac{d}{dt}\,\mathrm{x}(t) &= \mathrm{x}(t) \\ \frac{d}{dt}\,\mathrm{y}(t) &= -4\,\mathrm{x}(t) \\ \mathrm{x}(0) &= 1, \quad \mathrm{y}(0) = 0 \end{aligned}$$

as can easily be verified. A plot of this curve is given by

```
> plot([cos(2*t),-2*sin(2*t),t=0..Pi],scaling = constrained);
```

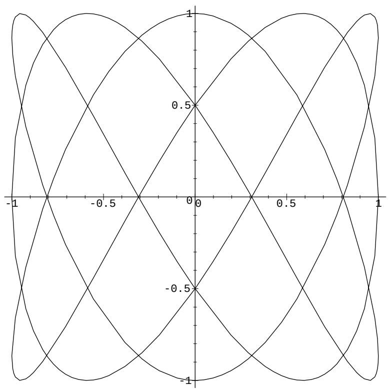


Figure 74: Lissajous Figure x = cos(3t), y = sin(5t)

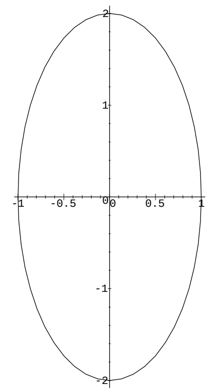


Figure 75: Ellipse x= cos(2t), y= -2sin(2t)

See Figure 75. When plotting curves in three dimensions using Maple V we need to use the command **spacecurve** which is part of the **plots** package. Consider the curve written parametrically as

$$x = \cos t, \;\; y = \sin t, \;\; z = t.$$

It's plot, Figure 76 over the range $[0, 2\pi]$ is given by

```
> spacecurve([cos(t),sin(t),t],t=0..2*Pi);
```

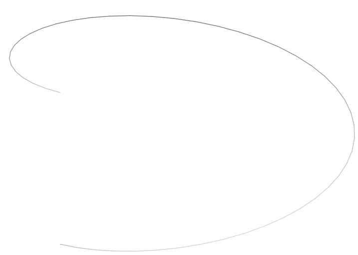

Figure 76: Cylindrical Helix: x = cos(t), y = sin(t), z= t

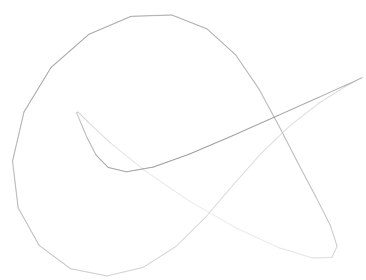

Figure 77: A Closed Curve in Three Space

The complicated curve shown in Figure 77, is actually a simple closed curve, and can be obtained as follows:

```
> spacecurve([(4+sin(3*t))*cos(2*t),(4+sin(3*t))*sin(2*t),cos(3*t)],
> t=0..2*Pi);
```

**Exercises 14.1**

1. Make a Maple V plot of the conical helix given by the parametric equations

$$x = t\cos(3t), \;\; y = t\sin(3t), \;\; z = t, \quad -\infty < t < \infty.$$

2. Write parametric equations for first octant portion of the curve of intersection of the sphere and cylinder

$$x^2 + y^2 + z^2 = 64, \quad x^2 + (y-4)^2 = 16.$$

Make a Maple V plot of this curve.

3. Show that $x = \sin(2t)\cos(t),\ y = \sin(2t)\sin(t), \quad 0 \le t \le 2\pi$ is a parametric representation for the curve given implicitly by the equation

$$(x^2 + y^2)^3 = 4x^2y^2.$$

Plot the graph of this curve using Maple V using the parametric representation and also using the explicit repesentation. Is it worthwhile to obtain the parametric representation?

## 14.2 Parametric Equations of Surfaces

As with the implicit representation of a circle, the implicit representation of a sphere may require a piece meal approach when making plots. In order to plot the graph of the sphere

$$x^2 + y^2 + z^2 = 4$$

we could solve for $z$ and obtain a parameterization for the lower half with

$$z = f_1(x, y) = -\sqrt{4 - x^2 - y^2}, \quad 0 \le x^2 + y^2 \le 4$$

and obtain a plot with **plot3d**.

```
> P1 := plot3d(-sqrt(4-x^2-y^2),x=-2..2,y=-sqrt(4-x^2)..sqrt(4-x^2),
> style=patchnogrid,numpoints=2500):";
```

In order to get the top half we type in the following:

```
> P2 := plot3d(sqrt(4-x^2-y^2),x=-2..2,y=-sqrt(4-x^2)..sqrt(4-x^2),
> style=patchnogrid,numpoints=2500):";
```

Finally, to put both plots, as shown in Figure 78, at the same time we write.

```
> with(plots):
> display({P1,P2},axes=normal,scaling=constrained);
```

```
> with(plots):
> display({P1,P2},scaling =constrained);
```

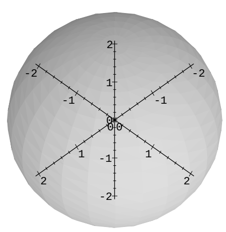

Figure 78: Sphere of Radius 2

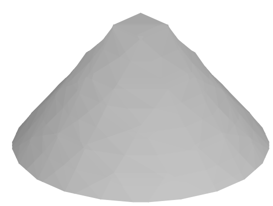

Figure 79: Cone Plot with **implicitplot3d**

This is analogous to the procedure that was use in plotting a circle when its implicit equation is given in rectangular coordinates. Just as with the situation with a circle, there is a parameterization which enables us to plot the graph with just one set of defining equations. From a well known trigonometric identity you can see that

$$x = 2\sin(u)\cos(v), \; y = 2\sin(u)\sin(v), \; z = 2\cos(u)$$

provides a parametric representation for the entire sphere if $0 \le u \le \pi, \; 0 \le v \le 2\pi$. The sphere can thus be plotted, as is also shown in Figure 79, as follows:

```
> plot3d([2*sin(u)*cos(v),2*sin(u)*sin(v),2*cos(u)],u=0..Pi,v=Pi..2*Pi,
> scaling=constrained,axes=normal,style=patch);
```

**Example:** Find a parameterizations for the bottom half of the cone that intersects the $z = 0$ plane in the circle

$$x^2 + y^2 = 4$$

and whose vertex is located a height 8 above the plane z=0.

**Solution:** An implicit equation for the entire cone is seen to be

$$4\left(1 - \frac{1}{8}z\right)^2 = x^2 + y^2$$

and thus the following use of **implicitplot3d** produces the plot, shown in Figure 79, of the surface.

```
> implicitplot3d(4*(1-z/8)^2=x^2+y^2,x=-3..3,y=-3..3,z=-1..17);
```

You can verify that if

$$x = \left(1 - \frac{1}{8}u\right)\cos(v),\ y = \left(1 - \frac{1}{8}u\right)\sin(v),\ z = u$$

then the point $(x, y, z)$ lies on the cone. Thus the following command plots the bottom half of the cone.

```
> plot3d([(1-u/8)*cos(v),(1-u/8)*sin(v),u],u=0..8,v=0..2*Pi);
```

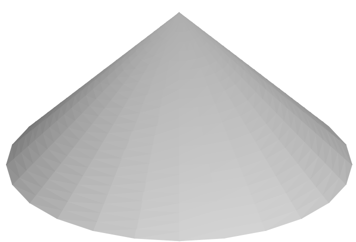

Figure 80: Cone Using Parametric Equations

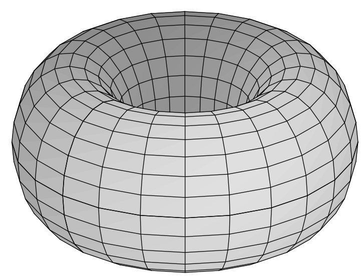

Figure 81: Plot of Torus

**Example:** Plot the torus which is created by revolving the circle

$$(x - 1)^2 + z^2 = 4$$

about the $z-$axis. After verifying that the following gives a parameterization for this torus

$$x = 2\cos(u) + \cos(v)\cos(u),\ y = 2\sin(u) + \cos(v)\sin(u),\ z = \sin(v)$$

it is easy to see that the the next command plots it. See Figure 81.

```
> plot3d([2*cos(u)+cos(v)*cos(u),2*sin(u)+cos(v)*sin(u),sin(v)],
> u=0..2*Pi,v=0..2*Pi);
```

**Exercises 14.2**

1. Find an implicit equation in rectangular coordinates for the surface given by

$$x = 2u\cos v,\ \ y = 3u\sin v,\ \ z = u.$$

   Make a Maple V plot of the surface using the given parametric representation.

2. Write parametric equations for the hyperboloid of one sheet

$$x^2 + y^2 = z^2 + 1$$

   and make a Maple V plot that illustrates the qualitative features of this surface.

## 14.3 Velocity and Acceleration Vectors in Space

Consider a curve parameterized by $t \in [a, b]$ :

$$x = f(t), \ y = g(t), \ z = h(t).$$

```
> with(linalg):

Warning: new definition for   norm
Warning: new definition for   trace
```

The position vector $R$ for this function is given by

```
> R := vector([f(t),g(t),h(t)]);
```

$$R := [\, \mathrm{f}(\,t\,)\, \mathrm{g}(\,t\,)\, \mathrm{h}(\,t\,)\,]$$

The velocity vector $V$ is obtained by differentiation, which since $R$ is a vector requires the **map** command.

```
> V := map(diff,R,t);
```

$$V := \left[ \frac{\partial}{\partial t}\mathrm{f}(\,t\,)\, \frac{\partial}{\partial t}\mathrm{g}(\,t\,)\, \frac{\partial}{\partial t}\mathrm{h}(\,t\,) \right]$$

The magnitude of the the velocity vector is the speed $\frac{ds}{dt}$.

```
> dsdt  := sqrt(dotprod(V,V));
```

$$dsdt := \left( \left(\frac{\partial}{\partial t}\mathrm{f}(\,t\,)\right)^2 + \left(\frac{\partial}{\partial t}\mathrm{g}(\,t\,)\right)^2 + \left(\frac{\partial}{\partial t}\mathrm{h}(\,t\,)\right)^2 \right)^{1/2}$$

The arc length of the curve is found through integrating the speed which gives the formula:

```
> Length := Int(s,t=a..b);
```

$$Length := \int_a^b \left( \left(\frac{\partial}{\partial t}\mathrm{f}(\,t\,)\right)^2 + \left(\frac{\partial}{\partial t}\mathrm{g}(\,t\,)\right)^2 + \left(\frac{\partial}{\partial t}\mathrm{h}(\,t\,)\right)^2 \right)^{1/2} dt$$

As an example consider the curve parameterized by

$$x = (4 + \sin t)\cos 3t, \ y = (4 + \sin t), \ z = \sin 3t, cos(t)$$

The position vector $R$ is

```
> R := vector([(4+sin(t))*cos(3*t),(4+sin(t))*sin(3*t),cos(t)]);
```

$$R := [\,(\,4 + \sin(\,t\,)\,)\cos(\,3\,t\,)\,(\,4 + \sin(\,t\,)\,)\sin(\,3\,t\,)\cos(\,t\,)\,]$$

A plot of this curve from $t = 0$ to $t = \pi$ is shown in Figure 82 and is produced by

```
> with(plots):
> spacecurve(R,t=0..2*Pi);
```

Thus the tangent vector is:

```
> V := map(diff,R,t);
```

$$V := [\cos(t)\cos(3t) - 3(4+\sin(t))\sin(3t)$$
$$\cos(t)\sin(3t) + 3(4+\sin(t))\cos(3t)\ -\sin(t)]$$

The square of the speed is

```
> dsdt2 := dotprod(V,V);
```

$$dsdt2 := (\cos(t)\cos(3t) - 12\sin(3t) - 3\sin(3t)\sin(t))^2$$
$$+ (\cos(t)\sin(3t) + 12\cos(3t) + 3\cos(3t)\sin(t))^2 + \sin(t)^2$$

```
> dsdt2 := simplify(dsdt2);
```

$$dsdt2 := -9\cos(t)^2 + 72\sin(t) + 154$$

The speed is

```
> dsdt := sqrt(dsdt2);
```

$$dsdt := \sqrt{-9\cos(t)^2 + 72\sin(t) + 154}$$

The length of the arc from $t = 0$ to $t = 2\pi$ is

```
> Length := Int(s,t=0..2*Pi);
```

$$Length := \int_0^{2\pi} \sqrt{-9\cos(t)^2 + 72\sin(t) + 154}\, dt$$

A ten digit approximation for this is given by

```
> Length := evalf(Length);
```

$$Length := 75.66806064$$

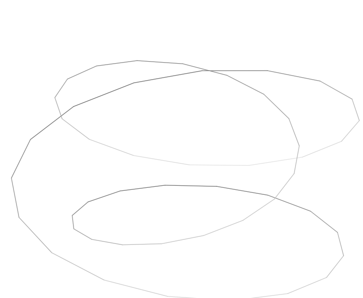

Figure 82: Curve in Space

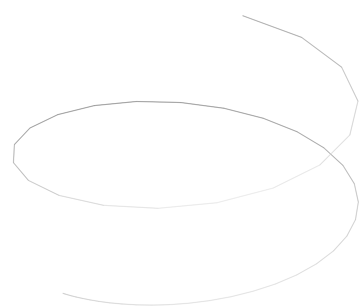

Figure 83: A Non-linear Helix

We now return our expressions $R$, $V$, and $\frac{ds}{dt}$ to the general case to discuss acceleration.

```
> R := vector([f(t),g(t),h(t)]); V:= map(diff,R,t); dsdt := sqrt(dotprod(V,V));
```

$$R := [\,\mathrm{f}(t)\ \mathrm{g}(t)\ \mathrm{h}(t)\,]$$

$$V := \left[\frac{\partial}{\partial t}\mathrm{f}(t)\ \frac{\partial}{\partial t}\mathrm{g}(t)\ \frac{\partial}{\partial t}\mathrm{h}(t)\right]$$

$$dsdt := \sqrt{\left(\frac{\partial}{\partial t}\mathrm{f}(t)\right)^2 + \left(\frac{\partial}{\partial t}\mathrm{g}(t)\right)^2 + \left(\frac{\partial}{\partial t}\mathrm{h}(t)\right)^2}$$

The unit tangent vector $T$ is the normalized tangent vector:

```
> T := scalarmul(V,1/dsdt);
```

$$T := \left[\frac{\frac{\partial}{\partial t}\mathrm{f}(t)}{\sqrt{\%1}}\ \frac{\frac{\partial}{\partial t}\mathrm{g}(t)}{\sqrt{\%1}}\ \frac{\frac{\partial}{\partial t}\mathrm{h}(t)}{\sqrt{\%1}}\right]$$
$$\%1 := \left(\frac{\partial}{\partial t}\mathrm{f}(t)\right)^2 + \left(\frac{\partial}{\partial t}\mathrm{g}(t)\right)^2 + \left(\frac{\partial}{\partial t}\mathrm{h}(t)\right)^2$$

One important fact about $T$ and its derivative is that they are orthogonal. To see this calculate the derivative.

```
> dTdt := map(diff,T,t);
```

$$dTdt := \left[-\frac{1}{2}\frac{\left(\frac{\partial}{\partial t}\mathrm{f}(t)\right)\%5}{\%2^{3/2}} + \frac{\%4}{\sqrt{\%2}}\ \ -\frac{1}{2}\frac{\left(\frac{\partial}{\partial t}\mathrm{g}(t)\right)\%5}{\%2^{3/2}} + \frac{\%3}{\sqrt{\%2}}\right.$$
$$\left. -\frac{1}{2}\frac{\left(\frac{\partial}{\partial t}\mathrm{h}(t)\right)\%5}{\%2^{3/2}} + \frac{\%1}{\sqrt{\%2}}\right]$$
$$\%1 := \frac{\partial^2}{\partial t^2}\mathrm{h}(t)$$
$$\%2 := \left(\frac{\partial}{\partial t}\mathrm{f}(t)\right)^2 + \left(\frac{\partial}{\partial t}\mathrm{g}(t)\right)^2 + \left(\frac{\partial}{\partial t}\mathrm{h}(t)\right)^2$$
$$\%3 := \frac{\partial^2}{\partial t^2}\mathrm{g}(t)$$
$$\%4 := \frac{\partial^2}{\partial t^2}\mathrm{f}(t)$$
$$\%5 := 2\left(\frac{\partial}{\partial t}\mathrm{f}(t)\right)\%4 + 2\left(\frac{\partial}{\partial t}\mathrm{g}(t)\right)\%3 + 2\left(\frac{\partial}{\partial t}\mathrm{h}(t)\right)\%1$$

Now take the dot product of T and its derivative:

```
> simplify(dotprod(T,dTdt));
```

$$0$$

Thus $T$ is perpendicular to its derivative.
The acceleration vector is equal to the derivative of the velocity vector:

```
> A := map(diff,V,t);
```

$$A := \left[ \frac{\partial^2}{\partial t^2} \mathrm{f}(t) \, \frac{\partial^2}{\partial t^2} \mathrm{g}(t) \, \frac{\partial^2}{\partial t^2} \mathrm{h}(t) \right]$$

The radius of curvature is given by

```
> r := s/sqrt(dotprod(dTdt,dTdt)):
```

The curvature is the reciprocal of the radius of curvature:

```
> kappa := 1/r:
```

**Example:** Find the tangential and normal components of acceleration and the curvature for the helix given by the equations for $t > 0$ :

$$x = 3\cos(t^2),\; y = 3\sin(t^2),\; z = t^2$$

**Solution:** The position vector is

```
> R := vector([3*cos(t^2),3*sin(t^2),t^2]);
```

$$R := [\, 3\cos(t^2) \; 3\sin(t^2) \; t^2 \,]$$

We aren't asked to plot this curve for this problem but we will anyway. See Figure 83

```
> plots[spacecurve](R,t=0..Pi);
```

The velocity and acceleration vectors are:

```
> V := map(diff,R,t); A := map(diff,V,t);
```

$$V := [\, -6\sin(t^2)\,t \; 6\cos(t^2)\,t \; 2\,t \,]$$

$$A := [\, -12\cos(t^2)\,t^2 - 6\sin(t^2) \;\; -12\sin(t^2)\,t^2 + 6\cos(t^2) \; 2 \,]$$

The speed and unit tangent vectors are

```
> dsdt := simplify(sqrt(simplify(dotprod(V,V))),symbolic);
```

$$dsdt := 2\sqrt{10}\,t$$

```
> T := scalarmul(V,1/dsdt);
```

$$T := \left[ -\frac{3}{10}\sqrt{10}\sin(t^2) \; \frac{3}{10}\sqrt{10}\cos(t^2) \; \frac{1}{10}\sqrt{10} \right]$$

Using vector analysis we know that the tangential component of acceleration is

$$\mathrm{dotprod(A, T)} \cdot \mathrm{T}.$$

```
> Atan := scalarmul(T,dotprod(A,T));
```

$$Atan := \left[ -\frac{3}{10}\,\%1\sqrt{10}\sin(t^2) \; \frac{3}{10}\,\%1\sqrt{10}\cos(t^2) \; \frac{1}{10}\,\%1\sqrt{10} \right]$$

$$\%1 := -\frac{3}{10}\left(-12\cos(t^2)\,t^2 - 6\sin(t^2)\right)\sqrt{10}\sin(t^2) + \frac{3}{10}\left(-12\sin(t^2)\,t^2 + 6\cos(t^2)\right)\sqrt{10}\cos(t^2) + \frac{1}{5}\sqrt{10}$$

```
> simplify(%1);
```

$$2\sqrt{10}$$

```
> Atan := map(z->subs(%1=2*sqrt(10),z),Atan);
```

$$Atan := [\,-6\sin(\,t^2\,)\;6\cos(\,t^2\,)\;2\,]$$

```
> absAtan := sqrt(simplify(dotprod(Atan,Atan)));
```

$$absAtan := 2\sqrt{10}$$

Of course the other formula for this quantitiy is $\frac{d^2s}{dt^2}$ :

```
> diff(dsdt,t);
```

$$2\sqrt{10}$$

If you know the acceleration vector and its tangential component then by vector subtraction you can get the normal component of acceleration.

```
> Anormal := evalm(A-Atan);
```

$$Anormal := [\,-12\cos(\,t^2\,)\,t^2\;\;-12\sin(\,t^2\,)\,t^2\;0\,]$$

The magnitude of this component is thus

```
> simplify(sqrt(simplify(dotprod(Anormal,Anormal))),symbolic);
```

$$12\,t^2$$

This means that the curvature is calculated by

```
> kappa := "/dsdt^2;
```

$$\kappa := \frac{3}{10}$$

Another way to obtain the normal component is from the formula:

$$\frac{ds}{dt}\cdot \mathrm{dT/dt}.$$

```
> dTdt := map(diff,T,t);
```

$$dTdt := \left[-\frac{3}{5}\sqrt{10}\cos(\,t^2\,)\,t\;\;-\frac{3}{5}\sqrt{10}\sin(\,t^2\,)\,t\;0\right]$$

```
> scalarmul(dTdt,dsdt);
```

$$[\,-12\cos(\,t^2\,)\,t^2\;\;-12\sin(\,t^2\,)\,t^2\;0\,]$$

This agrees with the previous expression fo the normal component of acceleration.

**Exercises 14.3**

1. If
$$\mathbf{R}(t) = \sin(2t)\mathbf{i} + 3\cos(3t)\mathbf{j} + t^2\mathbf{k},$$
find **V**, **A**, $\frac{ds}{dt}$, and the approximate value of the arc length of the curve as $t$ varies form 0 to $2\pi$. Plot a the curve over this interval.

## 14.4 Vector Fields

There are several graphing procedures in Maple V that help you to study vector fields.

```
> with(plots):
```

Within the **plots** package there is the command **gradplot** that allows you to plot the gradient vector field for a function of two variables. For a plot of the gradient vector field of the scalar function $f(x, y) = x^2 + 2y^2$ see Figure 84.

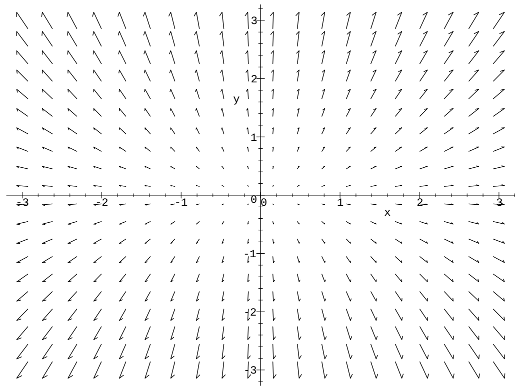

Figure 84: Gradient Plot of $x^2 + 2y^2$

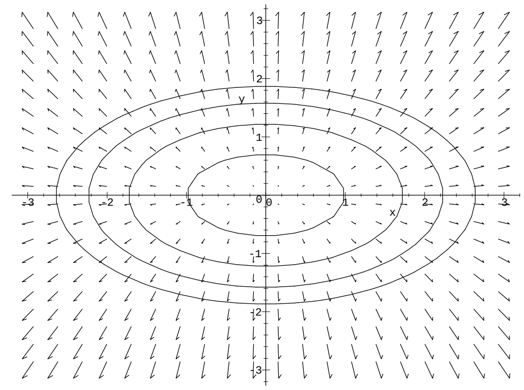

Figure 85: Gradient Plot with some Level Curves

```
> Q := gradplot(x^2+2*y^2,x=-3..3,y=-3..3):";
```

This command along with **implicitplot** allows you to plot the vector field along with several level curves. See Figure 85.

```
> P.1 := implicitplot(x^2+2*y^2=1,x=-3..3,y=-3..3):";
> for i from 1 to 3 do P.(2*i+1) := implicitplot(x^2+2*y^2=2*i+1,
> x=-3..3,y=-3..3):od:
> display({Q,seq(P.(2*i-1),i=1..4)});
```

If you wish to view the flow the vector field in two dimensions, then it is useful to utilize the tools that are part of the **DEtools** package.

```
> with(DEtools):
```

We first use **grad** to calculate the gradient vector field of the function.

```
> f := x^2+2*y^2;
```

$$f := x^2 + 2y^2$$

The **linalg** package needs to be made available for some of the commands that follow.

```
> with(linalg):

Warning: new definition for   norm
Warning: new definition for   trace

> X := grad(f,[x,y]);
```

$$X := [\,2\,x\;4\,y\,]$$

However, to use this vector field as an argument for **DEplot2** we must convert it to the type **list**.

```
> X := convert(X,list);
```

$$X := [\,2\,x,\,4\,y\,]$$

The next command plots the gradient vector field along with the three flowlines which start at the points $(-1.5, 1.1)$, $(1.1, 0.83)$, and $(0.8, 0.9)$. See Figure 86.

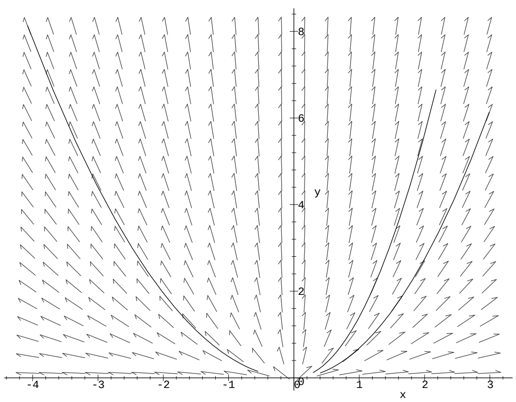

Figure 86: Gradient Vector Field and some Flow Lines

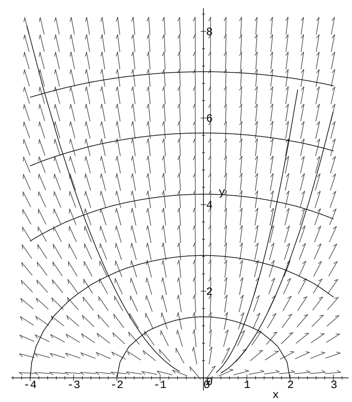

Figure 87: Gradient Vector Field with Level Curves and Flow Lines

```
> Flow := DEplot2(X,[x,y],-0.5..0.5,
> {[0,-1.5,1.1],[0,1.1,0.83],[0,0.8,0.9]}):";
```

Now if we want some level curves for $f(x, y)$, along with the vector field and flowlines, we do something like the following. See Figure 87.

```
> P.1 := implicitplot(x^2+2*y^2=1,x=-4..3,y=0..9):";
> for i from 1 to 5 do P.(2*i) := implicitplot(x^2+2*y^2=(2*i)^2,
> x=-4..3,y=0..9):od:
> display({Flow,seq(P.(2*i),i=1..5)},scaling=constrained);
```

It is also possible to plot two and three dimensional vector fields using **fieldplot** and **fieldplot3d**. For example to plot in Figure 88 of the vector field

$$< 2x,\ y,\ x - z >$$

execute the following commands:

```
> X := vector([2*x,y,x-z]);
```

$$X := [\,2\,x\,y\,x - z\,]$$

```
> fieldplot3d(X,x=-3..3,y=-3..3,z=-3..3);
```

**Exercises 14.4**

1. Plot the level curves
$$f(x, y) = x^2 - xy + y^2 = j,\ \text{for } j = 1, 2, 3$$
on a single Maple V plot with $-4 \le x \le 4$, $-4 \le y \le 4$. Make a plot of the gradient vector field along with flow lines through the points $(-1.5, 1)$, $(1, 1)$, $(1, 2)$, and $(1, -3)$.

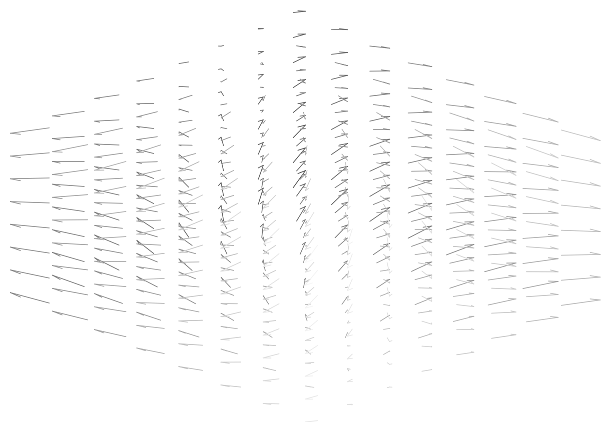

Figure 88: A Three Dimensional Field Plot

2. Plot flow lines for the vector field
$$\mathbf{F}(x, y) = < -2y,\ x >$$
that go through points $(1, 0)$, $(2, 0)$, and $(3, 0)$.

## 14.5 Line Integrals with Maple V

In this section we compute line integrals of vector fields over curves in two and three dimensional space.

```
> with(linalg):
```

For a start lets compute the line integral of the vector field $< x^2, \ x >$ over the straight line that connects (0,0) to (1,0).

```
> V := vector([x^2,x]);
```

$$V := [\, x^2 \; x \,]$$

```
> r := vector([t,0]);
```

$$r := [\, t \; 0 \,]$$

```
> dr := map(diff,r,t);
```

$$dr := [\, 1 \; 0 \,]$$

```
> B := subs({x=t,y=0},dotprod(V,dr));
```

$$B := t^2$$

```
> Int(V,r) = int(B,t=0..1);
```

$$\int V\,dr = \frac{1}{3}$$

As another example let's integrate the vector field

$$X = < xy, yz, xz >$$

over the twisted cubic, which is given parametrically by

$$x = t, \ \ y = t^2, \ \ z = t^3,$$

from $(-1, 1, -1)$ to $(1, 1, 1)$.

```
> V := vector([x*y,y*z,x*z]); r := vector([t,t^2,t^3]);
```

$$V := [\, x\,y \; y\,z \; x\,z \,]$$

$$r := [\, t \; t^2 \; t^3 \,]$$

```
> dr := map(diff,r,t);
```

$$dr := [\, 1 \; 2\,t \; 3\,t^2 \,]$$

```
> B := subs({x=t,y=t^2,z=t^3},dotprod(V,dr));
```

$$B := t^3 + 5\,t^6$$

```
> Int(V,r) = int(B,t=-1..1);
```

$$\int V \, dr = \frac{10}{7}$$

**Exercises 14.5**

1. Let $\mathbf{F}(x, y, z) =< x^3y,\ xy,\ yz >$ . Find the work done by $\mathbf{F}$ along a circle of radius 5 centered at the origin in the $yz$ plane.

2. Evaluate the line integral of the vector field $\mathbf{F}(x, y, z) =< xye^{-y},\ xz,\ \sin(z) >$ along the parabola $z = x^2,\ y = 0$, between $(-1, 0, 1)$ and $(1, 1, 1)$.

## 14.6 Divergence and Curl in Maple V

We can compute the curl and divergence of a vector field by using commands that are contained in the **linalg** package.

```
> with(linalg):

Warning: new definition for   norm
Warning: new definition for   trace
```

As an example we will calculate the divergence and curl of the vector field

$$< \sin x,\ \exp xyz,\ x + \sin z > .$$

```
> F := vector([sin(x),exp(x*y*z),x+sin(z)]);
```

$$F := [\sin(x)\ \ e^{(xyz)}\ \ x + \sin(z)]$$

```
> divF := diverge(F,[x,y,z]);
```

$$divF := \cos(x) + x\,z\,e^{(xyz)} + \cos(z)$$

```
> curlF := curl(F,[x,y,z]);
```

$$curlF := [-x\,y\,e^{(xyz)}\ \ -1\ \ y\,z\,e^{(xyz)}]$$

If we need to find the value of these quantities at a particular point, say $(0, 1, \pi)$, then we can use **subs** and **eval**.

```
> ValuedivF := subs(x=0,y=1,z=Pi,divF);
```

$$ValuedivF := \cos(0) + \cos(\pi)$$

```
> ValuedivF := simplify(ValuedivF);
```

$$ValuedivF := 0$$

```
> ValuecurlF := subs(x=0,y=1,z=Pi,eval(curlF));
```

$$ValuecurlF := [0\ \ -1\ \ \pi\,e^{0}]$$

```
> ValuecurlF := map(simplify,ValuecurlF);
```

$$ValuecurlF := [0\ \ -1\ \ \pi]$$

Note that there was a little difference in finding these values. The divergence is a scalar expression and so a simple use of **simplify** was able to simplify it. On the other hand the curl of a vector field is a vector and required the use of **eval** and **map** to simplify it completely.

**Exercises 14.6**

1. Compute the divergence and curl for the vector field

$$\mathbf{F}(x, y, z) = < x^2 \sin(yz),\ y^2 - xz,\ 2x^2 ze^{z^2} ).$$

   Evaluate the divergence and curl at the point (1,2,-1).

## 14.7 Surface Integrals with Maple V

Maple V can be useful in calculating surface integrals of vector fields over a parameterized surface. Let

$$\mathbf{F}(x,y,z) = < f(x,y,z),\ g(x,y,z),\ h(x,y,z) >$$

denote an arbitrary vector field in three dimensional space. We will define it to a Maple V session in the following. We assume that a surface $S$ is parameterized with

$$\mathbf{R}(u,v) = < x(u,v),\ y(u,v),\ z(u,v) > .$$

```
> f := 'f':g:= 'g':h:='h':
```

```
> with(linalg):
Warning: new definition for   norm
Warning: new definition for   trace
```

```
> F := vector([f(x,y,z),g(x,y,z),h(x,y,z)]);
```

$$F := [\,\mathrm{f}(x,y,z)\,\mathrm{g}(x,y,z)\,\mathrm{h}(x,y,z)\,]$$

The surface $S$ is defined by:

```
> R := vector([x(u,v),y(u,v),z(u,v)]);
```

$$R := [\,\mathrm{x}(u,v)\,\mathrm{y}(u,v)\,\mathrm{z}(u,v)\,]$$

The tangent vectors to the level curves $v =$ constant and $u =$ constant are computed as follows:

```
> Ru := map(diff,R,u);
```

$$Ru := \left[ \frac{\partial}{\partial u}\mathrm{x}(u,v)\ \frac{\partial}{\partial u}\mathrm{y}(u,v)\ \frac{\partial}{\partial u}\mathrm{z}(u,v) \right]$$

```
> Rv := map(diff,R,v);
```

$$Rv := \left[ \frac{\partial}{\partial v}\mathrm{x}(u,v)\ \frac{\partial}{\partial v}\mathrm{y}(u,v)\ \frac{\partial}{\partial v}\mathrm{z}(u,v) \right]$$

The normal, **NS**, to the surface for the given paramterization is defined by the following.

```
> NS := crossprod(Ru,Rv);
```

$$\begin{aligned} NS := \Bigg[ &\left(\frac{\partial}{\partial u}\mathrm{y}(u,v)\right)\left(\frac{\partial}{\partial v}\mathrm{z}(u,v)\right) - \left(\frac{\partial}{\partial u}\mathrm{z}(u,v)\right)\left(\frac{\partial}{\partial v}\mathrm{y}(u,v)\right) \\ &\left(\frac{\partial}{\partial u}\mathrm{z}(u,v)\right)\left(\frac{\partial}{\partial v}\mathrm{x}(u,v)\right) - \left(\frac{\partial}{\partial u}\mathrm{x}(u,v)\right)\left(\frac{\partial}{\partial v}\mathrm{z}(u,v)\right) \\ &\left(\frac{\partial}{\partial u}\mathrm{x}(u,v)\right)\left(\frac{\partial}{\partial v}\mathrm{y}(u,v)\right) - \left(\frac{\partial}{\partial u}\mathrm{y}(u,v)\right)\left(\frac{\partial}{\partial v}\mathrm{x}(u,v)\right) \Bigg] \end{aligned}$$

The flux integral

$$\int_S \mathbf{F}\cdot\mathbf{NS}\,dA$$

for the vector field can now be computed

```
> Int(F,R=S..NULL) = Int(Int(subs(x=x(u,v),y=y(u,v),z=z(u,v),
> dotprod(F,NS)),u=a..b),v=c..d);
```

$$\int_S F\,dR = \int_c^d \int_a^b \mathrm{f}(\mathrm{x}(u,v),\mathrm{y}(u,v),\mathrm{z}(u,v)) \left( \left( \frac{\partial}{\partial u}\mathrm{y}(u,v)(u,v) \right) \left( \frac{\partial}{\partial v}\mathrm{z}(u,v)(u,v) \right) - \left( \frac{\partial}{\partial u}\mathrm{z}(u,v)(u,v) \right) \left( \frac{\partial}{\partial v}\mathrm{y}(u,v)(u,v) \right) \right) + \mathrm{g}(\mathrm{x}(u,v),\mathrm{y}(u,v),\mathrm{z}(u,v)) \left( \left( \frac{\partial}{\partial u}\mathrm{z}(u,v)(u,v) \right) \left( \frac{\partial}{\partial v}\mathrm{x}(u,v)(u,v) \right) - \left( \frac{\partial}{\partial u}\mathrm{x}(u,v)(u,v) \right) \left( \frac{\partial}{\partial v}\mathrm{z}(u,v)(u,v) \right) \right) + \mathrm{h}(\mathrm{x}(u,v),\mathrm{y}(u,v),\mathrm{z}(u,v)) \left( \left( \frac{\partial}{\partial u}\mathrm{x}(u,v)(u,v) \right) \left( \frac{\partial}{\partial v}\mathrm{y}(u,v)(u,v) \right) - \left( \frac{\partial}{\partial u}\mathrm{y}(u,v)(u,v) \right) \left( \frac{\partial}{\partial v}\mathrm{x}(u,v)(u,v) \right) \right) du dv$$

**Example:** Evaluate the surface integral of the vector field

$$F = < x, y, -2z >$$

over the upper half of the sphere

$$x^2 + y^2 + z^2 = a^2$$

**Solution:** One parameterization is given by

$$x = a\cos(u)\sin(v),\ y = a\sin(u)\sin(v),\ z = a\sin(v)$$

```
> with(linalg):
```

```
Warning: new definition for   norm Warning: new
definition for   trace
```

```
> F := vector([x,y,-2*z]);
```

$$F := [\,x\ y\ -2\,z\,]$$

```
> R := vector([a*cos(u)*cos(v),a*sin(u)*cos(v),a*sin(v)]);
```

$$R := [\,a\cos(u)\cos(v)\ a\sin(u)\cos(v)\ a\sin(v)\,]$$

```
> Ru := map(diff,R,u);
```

$$Ru := [\,-a\sin(u)\cos(v)\ a\cos(u)\cos(v)\ 0\,]$$

```
> Rv := map(diff,R,v);
```

$$Rv := [\,-a\cos(u)\sin(v)\ -a\sin(u)\sin(v)\ a\cos(v)\,]$$

```
> NS := crossprod(Ru,Rv);
```

$$NS := [a^2 \cos(u) \cos(v)^2\ a^2 \sin(u) \cos(v)^2$$
$$a^2 \sin(u)^2 \cos(v) \sin(v) + a^2 \cos(u)^2 \cos(v) \sin(v)]$$

```
> NS := map(simplify,NS);
```

$$NS := [\, a^2 \cos(u) \cos(v)^2\ a^2 \sin(u) \cos(v)^2\ a^2 \cos(v) \sin(v)\,]$$

The integrand for the integral is

```
> H := subs(x=R[1],y=R[2],z=R[3],dotprod(NS,F));
```

$$H := a^3 \cos(u)^2 \cos(v)^3 + a^3 \sin(u)^2 \cos(v)^3 - 2\,a^3 \cos(v) \sin(v)^2$$

```
> H := simplify(H);
```

$$H := 3\,a^3 \cos(v)^3 - 2\,a^3 \cos(v)$$

```
> int(int(H,u=0..Pi/2),v=0..2*Pi);
```

$$0$$

Thus the value of this surface integral is 0.
As another example consider the surface $S$ which is bounded below by

$$z = x^2 + y^2$$

and above by the paraboloid

$$z = 4 - x^2 - y^2.$$

```
> with(plots):
```

We define our surface with the expressions

```
> z:= 'z': y := 'y': x := 'x': bottom := x^2+y^2;
```

$$bottom := x^2 + y^2$$

```
> top := 4 -x^2-y^2;
```

$$top := 4 - x^2 - y^2$$

and plot the surface as shown in Figure 89

```
> B:= cylinderplot(sqrt(z),theta=0..2*Pi,z=0..2):";

> T := cylinderplot(sqrt(4-z),theta=0..2*Pi,z=2..4):";

> display3d({B,T},axes=NORMAL,style=PATCH,scaling=CONSTRAINED,

> orientation=[45,75]);
```

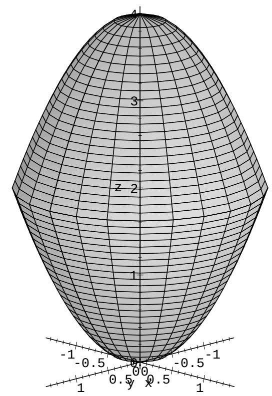

Figure 89: Figure Formed by Two Paraboloids

Let us assume that the density of the material that the surface is composed of is

$$\mathrm{d}(x, y, z) = \frac{z}{\pi} + \frac{1}{\pi}$$

and our problem is to calculate the total mass of the surface. This amounts to computing the surface integral of $d(x, y, z)$ over each of the two parts: the top and the bottom and adding the results.

```
> d := z/Pi+1/Pi;
```

$$d := \frac{z}{\pi} + \frac{1}{\pi}$$

Parameterizations for these surfaces are

```
> Rbottom := vector([u,v,u^2+v^2]);
```

$$Rbottom := [\, u\; v\; u^2 + v^2 \,]$$

```
> Rtop :=vector([u,v,4-u^2-v^2]);
```

$$Rtop := [\, u\; v\; 4 - u^2 - v^2 \,]$$

The normal for the bottom will now be computed.

```
> Rbottomu := map(diff,Rbottom,u); Rbottomv:= map(diff,Rbottom,v);
```

$$Rbottomu := [\, 1\; 0\; 2\,u \,]$$

$$Rbottomv := [\, 0\; 1\; 2\,v \,]$$

```
> Nbottom := crossprod(Rbottomu,Rbottomv);
```

$$Nbottom := [\,-2\,u \;\; -2\,v \;\; 1\,]$$

For this problem below we want Nbottom to have $z$ component pointing down so write:

```
> Nbottom := scalarmul(Nbottom,-1);
```

$$Nbottom := [\,2\,u \;\; 2\,v \;\; -1\,]$$

```
> magNBottom := sqrt(dotprod(Nbottom,Nbottom));
```

$$magNBottom := \sqrt{4\,u^2 + 4\,v^2 + 1}$$

The integrand is thus

```
> H := subs(x=u,y=v,z=u^2+v^2,d)*magNBottom;
```

$$H := \left(\frac{u^2 + v^2}{\pi} + \frac{1}{\pi}\right)\sqrt{4\,u^2 + 4\,v^2 + 1}$$

It follows that a double integral which is equal to the mass of the bottom surface is given by the following:

```
> Bmass := Int(Int(H,v=-sqrt(2-u^2)..sqrt(2-u^2)),u=-sqrt(2)..sqrt(2));
```

$$Bmass := \int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-u^2}}^{\sqrt{2-u^2}} \left(\frac{u^2 + v^2}{\pi} + \frac{1}{\pi}\right)\sqrt{4\,u^2 + 4\,v^2 + 1}\,dv\,du$$

This integral is pretty hard to evaluate even with Maple V

```
> value(Bmass);
```

$$\begin{aligned}\int_{-\sqrt{2}}^{\sqrt{2}} & \frac{3}{64}(24\,u^2\sqrt{2-u^2} + 24\ln(\,2\sqrt{2-u^2} + 3\,)\,u^2 + 66\sqrt{2-u^2} \\ & + 16\,u^4\ln(\,2\sqrt{2-u^2} + 3\,) + 5\ln(\,2\sqrt{2-u^2} + 3\,))\Big/\pi - \frac{3}{64}(-24\,u^2\sqrt{2-u^2} \\ & + 24\ln(\,-2\sqrt{2-u^2} + 3\,)\,u^2 - 66\sqrt{2-u^2} + 16\,u^4\ln(\,-2\sqrt{2-u^2} + 3\,) \\ & + 5\ln(\,-2\sqrt{2-u^2} + 3\,))\Big/\pi du\end{aligned}$$

Thus Maple V was unable to compute the exact value of this integral in its present form. But we can get a numerical approximation using **evalf**.

```
> evalf(");
```

$$9.300000000$$

If we use our head a little bit we can simplify the preceding double integral with polar coordinates, then $\cdots$

```
> Bmass := Int(Int((r^2+1)/Pi*sqrt(4*r^2+1)*r,r=0..sqrt(2)),
> theta=0..2*Pi);
```

$$Bmass := \int_0^{2\pi}\int_0^{\sqrt{2}} \frac{(\,r^2 + 1\,)\sqrt{4\,r^2 + 1}\,r}{\pi}\,dr\,d\theta$$

```
> Bmass := value(");
```

$$Bmass := \frac{93}{10}$$

Now we do the same thing for the top

```
> Rtopu := map(diff,Rtop,u); Rtopv := map(diff,Rtop,v);
```

$$Rtopu := [\,1\; 0\; -2\,u\,]$$

$$Rtopv := [\,0\; 1\; -2\,v\,]$$

```
> Ntop := crossprod(Rtopu,Rtopv);
```

$$Ntop := [\,2\,u\; 2\,v\; 1\,]$$

```
> magNtop := sqrt(dotprod(Ntop,Ntop));
```

$$magNtop := \sqrt{4\,u^2 + 4\,v^2 + 1}$$

```
> H := subs(x=u,y=v,z=4-u^2-v^2,d)*magNtop;
```

$$H := \left(\frac{4 - u^2 - v^2}{\pi} + \frac{1}{\pi}\right)\sqrt{4\,u^2 + 4\,v^2 + 1}$$

Converting to polar coordinates we have

```
> H := subs(u=r*cos(theta),v=r*sin(theta),H);
```

$$H := \left(\frac{4 - r^2\cos(\theta)^2 - r^2\sin(\theta)^2}{\pi} + \frac{1}{\pi}\right)\sqrt{4\,r^2\cos(\theta)^2 + 4\,r^2\sin(\theta)^2 + 1}$$

```
> H := simplify(H);
```

$$H := -\frac{\sqrt{4\,r^2+1}\,(-5+r^2)}{\pi}$$

```
> Tmass := Int(Int(H*r,r=0..sqrt(2)),theta=0..2*Pi);
```

$$Tmass := \int_0^{2\pi}\int_0^{\sqrt{2}} -\frac{\sqrt{4\,r^2+1}\,(-5+r^2)\,r}{\pi}\,dr\,d\theta$$

```
> Tmass := value(Tmass);
```

$$Tmass := \frac{167}{10}$$

```
> Total := Bmass + Tmass;
```

$$Total := 26$$

We conclude that the total mass of the surface is 26.

Let us continue with the surface that was defined in the last example and assume that we have a vector field

$$\mathbf{F} = < x,\ y,\ 1 >$$

given. Now the surface integral of this vector field over the bottom surface is

```
> F := vector([x,y,1]);
```

$$F := [\,x\,y\,1\,]$$

We now make use of the fact that we oriented the normal vector for the bottom surface.

```
> H := subs(x=u,y=v,z=sqrt(u^2+v^2),dotprod(F,Nbottom));
```

$$H := 2\,u^2 + 2\,v^2 - 1$$

```
> Bflux :=
> Int(Int(H,v=-sqrt(2-u^2)..sqrt(2-u^2)),u=-sqrt(2)..sqrt(2));
```

$$Bflux := \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-u^2}}^{\sqrt{2-u^2}} 2\,u^2 + 2\,v^2 - 1\,dv\,du$$

```
> Bflux := value(Bflux);
```

$$Bflux := 2\,\pi$$

Now we do the same thing using the top.

```
> H:= subs(x=u,y=v,z=sqrt(4-u^2-v^2),dotprod(F,Ntop));
```

$$H := 2\,u^2 + 2\,v^2 + 1$$

```
> Tflux := Int(Int(H,v=-sqrt(2-u^2)..sqrt(2-u^2)),u=-sqrt(2)..sqrt(2));
```

$$Tflux := \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-u^2}}^{\sqrt{2-u^2}} 2\,u^2 + 2\,v^2 + 1\,dv\,du$$

```
> Tflux := value(");
```

$$Tflux := 6\,\pi$$

```
> SurfaceFlux := Tflux+Bflux;
```

$$SurfaceFlux := 8\,\pi$$

We conclude that the flux through the surface due to our vector field is $8\pi$. Can we verify this value using the Divergence Theorem? We start by computing the divergence of $F$.

```
> divF := diverge(F,[x,y,z]);
```

$$divF := 2$$

```
> SurfaceFlux := Int(Int(Int(divF,z=bottom..top),y=-sqrt(2-x^2)..
> sqrt(2-x^2)),x=-sqrt(2)..sqrt(2));
```

$$SurfaceFlux := \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} 2\,dz\,dy\,dx$$

```
> SurfaceFlux := value(SurfaceFlux);
```

$$SurfaceFlux := 8\,\pi$$

This verifies Gauss's Theorem for this problem since both methods for computing the surface flux yield the same answer.

**Example:** Verify Stoke's Theorem for

$$F =< 2x - y, -yz^2, -y^2z >,$$

where $S$ is the upperhalf surface of the sphere

$$x^2 + y^2 + z^2 = 1$$

and $C$ is the boundary.

**Solution:** First we calculate the line integral of the vector field over $C$. The circle $C$ is given by

```
> R := vector([cos(t),sin(t),0]);
```

$$R := [\cos(t)\ \sin(t)\ 0]$$

We now prepare to find the value of the integral that occurs on the right hand side of Stoke's Theorem.

```
> dR := map(diff,R,t);
```

$$dR := [-\sin(t)\ \cos(t)\ 0]$$

```
> F := vector([2*x-y,-y*z^2,-y^2*z]);
```

$$F := [2\,x - y\ \ -y\,z^2\ \ -y^2\,z]$$

```
> H := subs(x=cos(t),y=sin(t),z=0,dotprod(F,dR));
```

$$H := -(2\cos(t) - \sin(t))\sin(t)$$

```
> RightSide := Int(H,t=0..2*Pi);
```

$$RightSide := \int_0^{2\pi} -(2\cos(t) - \sin(t))\sin(t)\,dt$$

```
> RightSide := value(RightSide);
```

$$RightSide := \pi$$

Now we calculate the value of the integral on the left hand side of the equality in Stoke's Theorem. First we obtain a parametrization for the upper hemisphere:

$$x = \cos(u)\cos(v),\ y = \sin(u)\cos(v),\ z = \sin(v)$$

```
> R:= vector([cos(u)*cos(v), sin(u)*cos(v),sin(v)]);
```

$$R := [\cos(u)\cos(v)\ \sin(u)\cos(v)\ \sin(v)]$$

```
> Ru := map(diff,R,u);
```

$$Ru := [-\sin(u)\cos(v)\ \cos(u)\cos(v)\ 0]$$

```
> Rv := map(diff,R,v);
```

$$Rv := [-\cos(u)\sin(v)\ \ -\sin(u)\sin(v)\ \cos(v)]$$

```
> NS := map(simplify,eval(crossprod(Ru,Rv)));
```

$$NS := [\cos(u)\cos(v)^2\ \sin(u)\cos(v)^2\ \cos(v)\sin(v)]$$

```
> curlF := curl(F,[x,y,z]);
```

$$curlF := [0\,0\,1]$$

```
> H:= subs(x=cos(u)*cos(v),y=sin(u)*cos(v),z=sin(v),dotprod(curlF,NS));
```

$$H := \cos(v)\sin(v)$$

```
> LeftHand := Int(Int(H,u=0..2*Pi),v=0..Pi/2);
```

$$LeftHand := \int_0^{1/2\,\pi}\int_0^{2\pi}\cos(v)\sin(v)\,du\,dv$$

```
> LeftHand:= value(LeftHand);
```

$$LeftHand := \pi$$

Thus the two integrals have the same value as predicted by Stoke's Theorem.

**Exercises 14.7**

1. Let $\mathbf{F}(x, y, z) = < 2xy,\ 3yz,\ -4z^2 >$, and $S$ be the sphere $x^2 + y^2 + z^2 = 25$, oriented so that the normal points outward. Verify the Divergence Theorem in this case by evaluating the flux integral of the vector field $\mathbf{F}$ over the surface, and then by computing the triple integral of div $\mathbf{F}$ over the interior of $S$.

2. Compute the line integral of the vector field $\mathbf{F}(x, y, z) = < x^2,\ z - yx,\ xz >$ over the curve of intersection of the sphere $x^2 + y^2 + z^2 = 1$, with the plane $y = z$. Assume this curve is oriented so that its projection on the $xy$ plane is oriented clockwise. Calculate the integral directly and then use Stoke's Theorem to check your answer.