Project 2

Cables Hanging under their own Weight.

It is strongly recommended that you work Lab 6 before doing this project. This particular project does not use Maple V.

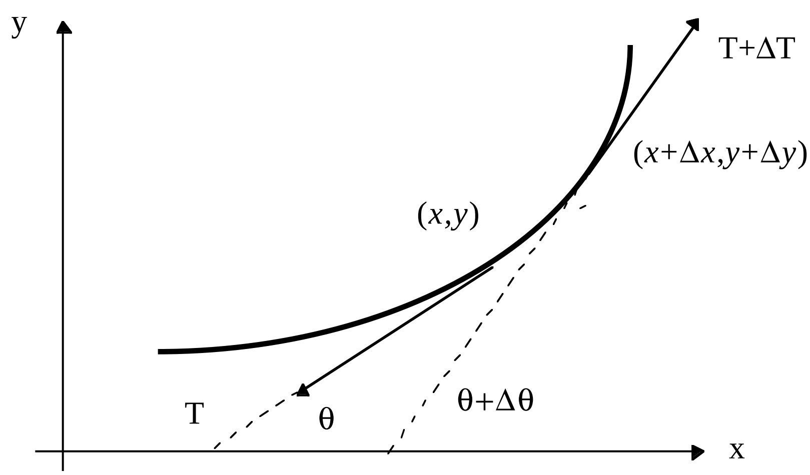
This project deals with examining cables, that is to say a line or rope which has a constant line density (the units will be grams/centimeter or pounds per foot or some other appropriate units) and whose ends are fixed. The object is to find the shape they assume. In the example we will look at a cable which is suspended from two distinct points which are at the same height. You should look review the notion of arc length which is on page 433 of your book.

Let's suppose that θ is some given angle and that $\Delta\theta$ is a small increment. Now use the tangent line approximations for sin and cos, see problem 7 on page 237, to derive

(a) $\cos(\theta+\Delta\theta)\approx\cos(\theta)-\Delta\theta\sin(\theta)$ and $\sin(\theta+\Delta\theta)\approx\sin(\theta)+\Delta\theta\cos(\theta)$.

Assume now that we have set up x and y axes so that x is horizontal and y is vertically upwards, and that the cable is represented by the equation $y=f(x)$. Our objective is to find $f(x)$.

We begin by deriving the equations of equilibrium of a small piece of the cable. We take a typical point (x,y) on the cable and a nearby point $(x+\Delta x, y+\Delta y)$.



(b) Show that $\dfrac{dy}{dx}=\tan(\theta)$, $\dfrac{dx}{ds}=\cos(\theta)$, $\dfrac{dy}{ds}=\sin(\theta)$.

Also show that $\dfrac{ds}{dx}=\sqrt{1+(\dfrac{dy}{dx})^2}$.

We now balance the horizontal forces:

$$T\cos(\theta) = (T + \Delta T)\cos(\theta + \Delta\theta),$$

(c) Use (a) to show that $\Delta T\cos(\theta) \approx T\sin(\theta)\Delta\theta.$

(d) Now let $\Delta\theta \to 0$ and show that $T\cos(\theta) = H$, where H is a constant. This says that the horizontal component of the tension in the cable is constant.

We now balance the vertical forces:

$$(T + \Delta T)\sin(\theta + \Delta\theta) = \omega\Delta s + T\sin(\theta),$$

where ω is the line density.

(e) Use (a) to show that $T\Delta\theta\cos(\theta) + \Delta T\sin(\theta) \approx \omega\Delta s$.

(f) Now let $\Delta\theta \to 0$ and show that $\frac{d}{ds}(T\sin(\theta)) = \omega.$

(g) We now eliminate T between (d) and (f) to derive

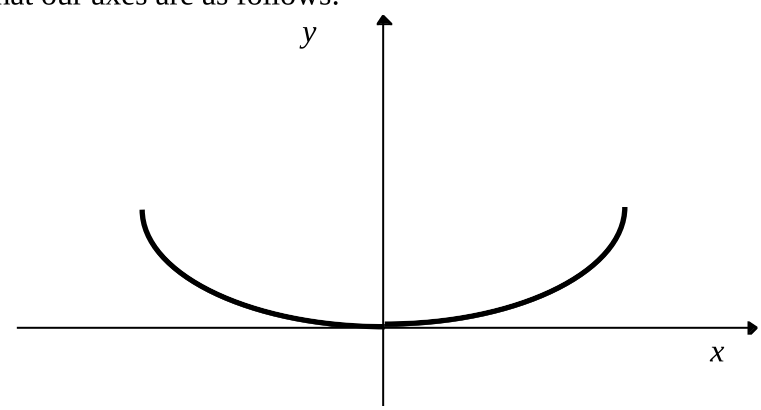
$$\frac{d}{ds}(\tan(\theta)) = \frac{\omega}{H},$$

and then use (b) to produce

(h)

$$\frac{d^2y}{dx^2} = \frac{\omega}{H}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ .$$

This is a nasty looking differential equation for the function y that we are looking for. However by making use of some hyperbolic functions we can actually solve it. We will suppose that our axes are as follows:



This means that we are setting $y(0) = 0, \quad y'(0) = 0.$

Then we set $\frac{dy}{dx} = \sinh z.$ Substituting this into (h) gives,

$$\frac{dz}{dx} = \frac{\omega}{H}.$$

This tells us that $z = \frac{\omega x}{H}$, so that

$$\frac{dy}{dx} = \sinh(\frac{\omega x}{H}).$$

Integrating this finally gives

(i)

$$y = \frac{H}{\omega}\cosh(\frac{\omega x}{H}) - \frac{H}{\omega}.$$

This curve is quite a famous one and is called a *catenary*.

(j) Now derive the following formulas for the tension in the cable,

$$T = H\cosh(\frac{\omega x}{H}) = H + \omega y.$$

(k) Now derive the following formulas for arc length measured from the origin,

$$s = \frac{H}{\omega}\sinh(\frac{\omega x}{H}) = \sqrt{y^2 + 2y\frac{H}{\omega}}.$$