9 Ordinary Differential Equations

Thoughout this book we have studied the two major concepts of calculus: the derivative, and the integral. In this Chapter we begin the study of differential equations. Consider a problem that we can solve right now. Suppose that we know the derivative of a function, $x(t)$, and the value of the function at one value of time, for example, suppose that it is given that

$$x'(t) = 2\,t - 2$$

$$x(0) = 2.$$

Can we find this function? In terms of mathematics, this is the same problem as the one studied in Chapter 6, where it was shown how to construct a function from a knowledge of its derivative. To find $x(t)$ one asks what function has a derivative equal to $2\,t-2$? It is possible to find all such functions by taking the antiderivative. Thus $x(t)$ has the form $x(t) = t^2 - 2\,t + C$, where C is a constant. Since the function $x(t)$ satisfies $x(0) = 2$, the constant C can be found from

$$2 = 0^2 - 2\,0 + C.$$

Hence a function $x(t)$ which satisfies the two equations above is $x(t) = t^2 - 2\,t + 2$. Since any other function, which satisfies the first equation must differ from $x(t)$ by only a constant, and since it also satisfies the second equation, it turns out that

$$x(t) = t^2 - 2\,t + 2$$

is the unique solution to the problem which was posed.

The problem of finding the antiderivative of a function is an example of a differential equation. Suppose that $f(t)$ is a continuous function defined over some interval $[a, b]$. A solution of the differential equation

$$x'(t) = f(t)$$

is an antiderivative of $f(t)$,

$$x(t) = \int f(t)\,dt + C.$$

Since whenever $x(t)$, defined in this way, is substituted for x into the equation $x' = f(t)$ equality holds, $x(t)$ is justifiably called a solution of the differential equation. In the next section we will define what a first order differential equation is and what is meant by a solution. The remainder of this chapter is concerned with some ways to solve differential equations and how differential equations can be used to solve many problems that arise in several fields of study.

9.1 What is a Differential Equation?

Water left in a glass cools or heats to the temperature of the surrounding air. If you drop an object into a body of water, then the object eventually approaches the the temperature of the water. These observations are examples of a general physical law called *Newton's Law of Cooling.* For example, let $T(t)$ be the temperature of the object at time t and T_s be the surrounding temperature, then according to Newton's Law of Cooling the rate of change of the temperature satisfies

$$\frac{dT}{dt} = -k(T - T_s),$$

where k is a constant that depends on the physical properties of the object. This is an example of a differential equation, where T is the dependent variable and t is the independent variable.

Definition
Let $f(t, x)$ be a function of two independent variables t and x. An equation of the form

$$x' = f(t, x)$$

is called a **first order differential equation**. The variable x is the *dependent variable* and the variable t is called the *independent variable.* A differentiable function $\phi(t)$ which is defined on some interval I and such that

$$\phi'(t) = f(t, \phi(t))$$

is satisfied for all $t \in I$ is called a **solution** of the differential equation.

For the differential equation that we derived from Newton's Law of Cooling the dependent variable was T instead of x and the function f in the definition is

$$f(t, T) = -k(T - T_s).$$

In the introductory section to this Chapter we saw how to solve a special type of differential equation, *i.e.* one in which the second variable x is absent in f. Consider a situation in which the variable t is absent, for example, suppose $f(t, x) = 2 - x$, *i.e.,* the differential equation is

$$x' = 2 - x.$$

Observe that this differential equation has the same form as the one derived from Newton's Law of Cooling, except that in this case x is used in place of T, $k = 1$, and $T_s = 2$. It will be shown how to to find a solution to this differential equation in Section 9.4. For the time being we will show how to determine if a given function is a solution. Let

$$\phi(t) = 2 - Ce^{-t},$$

where C is a constant. It will now be shown that $\phi(t)$ is a solution to the differential equation

$$x' = 2 - x.$$

In order to show that $\phi(t)$ is a solution, it must be verified that

$$\phi'(t) = 2 - \phi(t).$$

The left-hand side of the preceeding equality can be found by differentiating $\phi(t)$:

$$\text{Left} - \text{hand side } = \phi'(t) = Ce^{-t}.$$

The right-hand side is obtained by algebraic manipulation:

$$\text{Right} - \text{hand side } = 2 - \phi(t) = 2 - (2 - Ce^{-t}) = Ce^{-t}.$$

It follows that

$$\phi'(t) = 2 - \phi(t)$$

and, thus,

$$\phi(t) = 2 - Ce^{-t}$$

is a solution to the first order differential equation

$$x' = 2 - x.$$

Later in Section 9.4, you will learn how to solve this equation by hand, but now an illustration using Maple V to solve the equation will be given. The first step is to define the differential equation in a Maple V session.

```
> deq := diff(x(t),t)=2-x(t);
```

$$deq := \frac{d}{dt}x(t) = 2 - x(t)$$

Notice that when the equation was originally written the independent variable t was not shown explicitly. When defining differential equations for use with Maple V, one must express them in terms of the independent variable (t in this case), otherwise the Maple V **dsolve** command which is about to used will not treat things correctly. This command will be applied and some of its many options will be explored throughout this chapter, but for now the **dsolve** will be applied in its most basic form. Roughly speaking the **dsolve** command does for differential equations what **solve** and **fsolve** does for algebraic equations. The basic syntax is **dsolve**(diffeqn, vars), where *diffeqn* is a differential equation, and *vars* is the variables to be solved. In this case we have:

```
> dsolve(deq,x(t));
```

$$x(t) = 2 + e^{-t}\_C1$$

How does this result compare with the solution that was given above? Notice that if $C = -\_C1$, then there is no difference. Observe further that the answer is in the form of an equation. In order to manipulate the solution it is better to write it in terms of an expression or a function.

```
> phi := rhs(");
```

$$\phi := 2 + e^{-t}\_C1$$

This defines ϕ as an expression. If you wish the solution to be expressed as a function, then use **unapply**.

```
> phi := unapply(phi,t);
```

$$\phi := t \mapsto 2 + e^{-t}\_C1$$

You can verify that ϕ is a solution by having Maple V perform the same tasks that are done when checking it by hand. First calculate the left-hand side of the differential equation with ϕ substituted for x.

```
> LeftHandSide := diff(phi(t),t);
```

$$LeftHandSide := -e^{-t}\_C1$$

Next substitute $\phi(t)$ for $x(t)$ into the right-hand side of the equation.

```
> RightHandSide := 2 - phi(t);
```

$$RightHandSide := -e^{-t}\_C1\,t$$

Since the left- and right-hand sides agree, you may conclude that the function ϕ is a solution.

The constant C in the solution above is like the constants of integration that were encountered when finding antiderivatives. The only difference is that it enters the definition of ϕ as a multiplication factor; and the constants obtained from antidifferentiation are additive constants.

What is the significance of the constant? If one knows the value of the solution at one point then, just as with an antiderivative, one can determine the solution completely. For example suppose that it is required to find a solution of the differential equation

$$x' = 2 - x$$

which also satisfies the initial condition

$$x(0) = 3.$$

Then one can solve for the constant by solving the equation

$$3 = 2 - C\,e^{-0}$$

for C. This is easily solved by hand and it may be concluded that $C = -1$. Consequently, the solution is

$$\phi_1(t) = 2 + e^{-t}.$$

Using Maple V, proceed as follows. First solve for $\_C1$.

```
> C1 := solve(phi(0)=3,_C1);

                    C1 := 1
```

Then substitute the value into the solution that contains the arbitrary constant.

```
> phi1 := t -> subs(_C1=C1,phi(t));
```

$$phi1 := t \mapsto 2 + e^{-t}$$

You can plot your solution. See Figure 45

```
> plot(phi1(t),t=0..5);
```

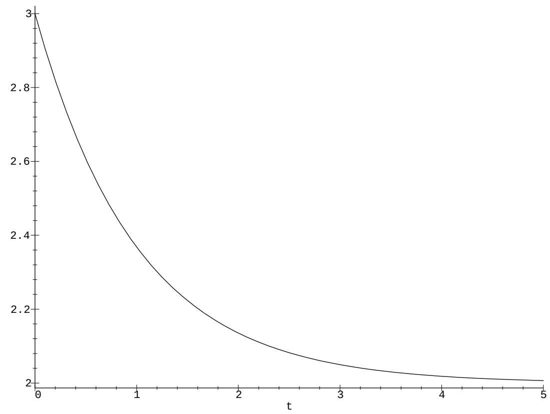

Figure 45: A solution curve

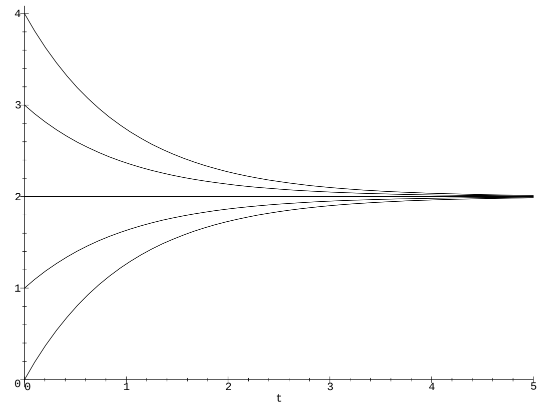

Figure 46: Several solution curves

Once you are fortunate enough to have a formula for the solution, you can use it to analyze its algebraic, geometric and numerical properties. For example, you can make multiple plots. See Figure 46 for plots of the solutions for $\_C1 = -2, -1, 0, 1, 2$.

```
> plt := [seq(plot(subs(_C1=i,phi(t)),t=0..5),i=-2..2)]:
> plots[display](plt);
```

Figure 46 suggests that regardless of the value that is assigned $\_C1$, every solution approaches the same horizontal asymptote $x = 2$.

```
> limit(phi(t),t=infinity);
```

2

It follows that every solution of the equation $x' = 2 - x$ approaches 2 asymptotically as $t \to \infty$. Recall that this differential equation is equivalent to one that arises from Newton's Law of Cooling when the constant $k = 1$ and $T_s = 2$. Thus the fact that all solutions approach $T_s = 2$ asymptotically is consistent with the statement that an object's temperature cools or heats to its surroundings.

Could we have anticipated this result before solving the equation? Suppose that $\phi(t)$ is a solution of the differential equation and $\phi(t) < 2$. Then the slope of the curve $x = \phi(t)$ satisfies

$$\phi'(t) = 2 - \phi(t) > 0.$$

This means that so long as $\phi(t) < 2$, that $\phi(t)$ is increasing. With the same reasoning it follows that $\phi(t)$ is decreasing whenever $\phi(t) > 2$. Moreover, when $\phi(t)$ is near 2, the slope of the curve $x = \phi(t)$ is almost 0. This suggests that, but certainly does not prove, that $\phi(t)$ approaches 2 as t gets large.

In the beginning of this section you were given a definition of a first order differential equation. You might wonder about the term *first order*. The order of a differential equation is equal to the highest order derivative that occurs in the equation.

Definition
Let $f(t, x, y)$ be a function of three variables t, x, and y. An equation of the form

$$x'' = f(t, x, x')$$

is called a **second order differential equation**. A differentiable function $\phi(t)$ which is defined on some interval I and such that

$$\phi''(t) = f(t, \phi(t), \phi'(t))$$

is satisfied for all $t \in I$ is called a **solution** of the differential equation.

As an example of a second order differential equation that arises in physical problems, consider a mass, m, that is attached to a spring. According to the physical law, known has Hooke's Law, the amount of force required to stretch or compress a spring is proportional to the length of the stretch or compression. If x denotes the displacement and k is the proportionality constant then the force F is

$$F(x) = k\,x.$$

Another law from physics, Newton's Second Law, states that

$$mx'' = -kx.$$

If we divide both sides by m, then

$$x'' = -\frac{k}{m}x,$$

so that

$$f(t, x, y) = -\frac{k}{m}x.$$

Because of Newton's Second Law second order differential equations tend to come up in many problems involving the motion of masses, and are thus of great interest.

As a particular example, of a second order differential equation consider

$$\frac{d^2x}{dt^2} + 2\,\frac{dx}{dt} + 25\,x = 0.$$

In this case

$$x'' = \frac{d^2x}{dt^2}$$

and

$$f(t, x, y) = -2\,y - 25\,x.$$

Differential equations like this can arise in many problems, for example, is spring-mass systems with resistance. It can be entered into a Maple V session as follows.

```
> eqn := diff(x(t),t$2)+2*diff(x(t),t)+25*x(t)=0;
```

$$eqn := \frac{d^2}{dt^2}x(t) + 2\,\frac{d}{dt}x(t) + 25\,x(t) = 0$$

It can be verified that

$$\phi(t) = e^{-t}\,\cos(2\sqrt{6}\,t)$$

is a solution of the differential equation by direct substitution and simplification.

```
> subs(x(t)=exp(-t)*cos(2*sqrt(6)*t),eqn);
```

$$\frac{d^2}{dt^2}(e^{-t}\ \cos(2\sqrt{6}\ t)) + 2\ \frac{d}{dt}(e^{-t}\ \cos(2\sqrt{6}\ t)) + 25\ (e^{-t}\ \cos(2\sqrt{6}\ t)) = 0$$

```
> simplify(");
```

```
0 = 0
```

Which shows that $\phi(t)$ is a solution. If you still feel that Maple V is not a useful tool, you might try verifying this by hand.

Exercises 9.1 In the following verify each $\phi(t)$ satisfies the indicated differential equation and plot $\phi(t)$.

1. Show that $\phi(t) = e^{3t}$ is a solution of the differential equation
$$x' = 3x.$$
2. Show that $\phi(t) = \sin(6t)$ is a solution of the differential equation
$$x'' + 36\ x = 0.$$
3. Show that $\phi(t) = -\frac{1}{t-5}$ is a solution of the differential equation
$$x' = x^2.$$
4. Show that $\phi(t) = \frac{100e^{t/10}}{1+e^{t/10}}$ is a solution of the differential equation
$$x' = 0.001x(100 - x).$$

9.2 Direction Fields

Let
$$x' = f(t, x)$$
be a first order differential equation. At each point in the (t, x)-plane where $f(t, x)$ is defined, the right-hand side of the equation gives a value of the derivative,
$$\phi'(t) = f(t, \phi(t)),$$
of the solution through that point. This derivative can be thought of as a the slope of a line segment through that point. The collection of all such line segments is called the *direction field*, (sometimes also called the *slope field*), for the differential equation. Maple V has a procedure that produces a plot of a direction field. The procedure is part of **DEtools** package and is called **DEplot1**. The syntax for using this procedure is **DEplot1**(deq,vars,trange,inits,xrange,options), where *deq* is the right-hand side of the first-order differential equation, *vars* is list of the variables that are used, *trange* is the range over which the independent variable ranges, *inits* is a set consisting of the initial conditions of the solutions which are to be plotted. If the *inits* is omitted, then only the direction field is drawn and no solutions are plotted. The variable *xrange* is the range over which the dependent variable ranges. If only solution curves are required then the *inits* must be non-empty, and the option **arrows** should be set equal to **NONE**.

Example 9.2.1 Use the Maple V to obtain the direction field for the differential equation
$$x' = 2 - x.$$
Then make a multiple plot of the solution curves to the differential equation which satisfy the five initial equations $x(0) = 0$, $x(0) = 1$, $x(0) = 2$, $x(0) = 3$, and $x(0) = 4$. Finally, make a plot which is a composite of the preceding plots.

Solution: Since the **DEtools** package is to be used, make the call using **with**. Then apply the procedure **DEplot1** to obtain the direction field shown in Figure 47. Use the option **arrows** = **LINE**.

```
> with(DEtools):
> plt1 := DEplot1( 2 - x,[t,x],t=0..5,x=-4..4,arrows = LINE):";
```

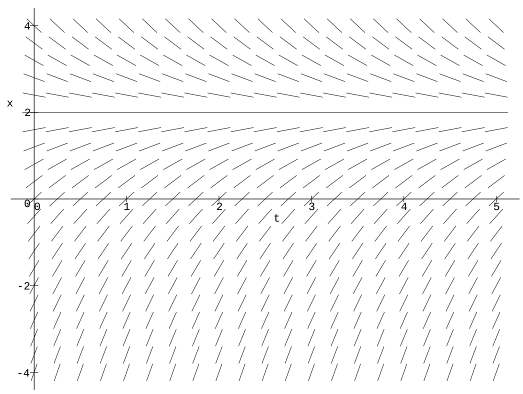

Figure 47: Direction field for $x' = 2 - x$

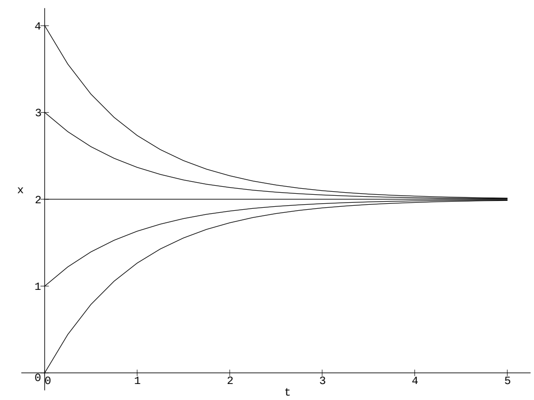

Figure 48: Five solution curves for $x' = 2 - x$

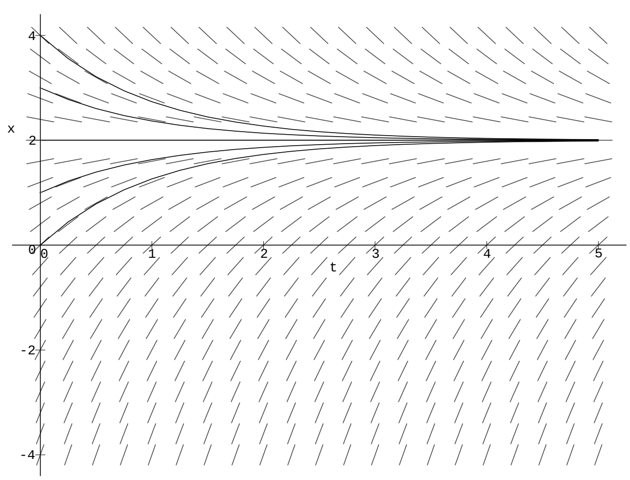

Figure 49: Solution curves and direction field

Now we plot the five solution curves corresponding to the solutions which satisfy the initial conditions:

$$x(0) = 0, x(0) = 1, x(0) = 2, x(1) = 3, \quad \text{and} \quad x(2) = 4.$$

In the previous section we made similar plots and we could use the same method that was used in there, but **DEplot1** will be used again. See Figure 48 and compare it with Figure 46.

```
> plt2 := DEplot1(2 - x,[t,x],t=0..5,{[0,0],[0,1],
> [0,2],[0,3],[0,4]}, arrows = NONE): ";
```

Figure 47 shows a collection of line segments through points in the (t, x)-plane. Each of these line segments is tangent to the solution curve of $x' = 2 - x$ through that point. A good way to see this is to combine this plot with some solution curves as those in Figure 48. See Figure 49.

```
> plots[display]({plt1,plt2});
```

Figure 48 can also be created by f a single call to the procedure **DEplot1**.

```
> DEplot1(diff(x(t),t)=2-x(t), [t,x],t=0..5,{[0,0],[0,1],[0,2],

> [0,3],[0,4]},arrows= LINE,x=-4..4);
```

Now consider a differential equation which comes up in population growth models called the *logistic equation*. Let $x(t)$ be the population of a certain species at time t. Assume a certain birth rate, kx, which causes the population to grow. If there is nothing to check this growth, then rate of growth of $x(t)$ satisfies the differential equation

$$x' = kx.$$

Assume $k > 0$, since we are assuming a birth rate, as opposed to death rate. In this special case growth takes the form

$$\phi(t) = Ce^{kt},$$

where C is a constant. These solutions tend to ∞ exponentially and soon would overpopulate the universe. More realistically there are factors that limit growth as a population increases. Thus for logistic growth assume that x' is also proportional to an expression of the form $1 - \frac{x}{M}$. Thus it is assumed that x satisfies a differential equation of the form

$$x' = kx(1 - \frac{x}{M}).$$

In the next example $k = 1$ and $M = 10$.

Example 9.2.2 Use the Maple V to obtain the direction field for the differential equation

$$x' = \frac{x(10 - x)}{10}.$$

Then make a multiple plot of the solution curves to the differential equation which satisfy the five initial equations $x(0) = 0$, $x(0) = 2$, $x(0) = 8$, $x(0) = 10$, $x(0) = 12$ and $x(0) = 20$. Finally, make a plot which is a composite of the preceding plots.

Solution We proceed as in the previous example. First call up the **DEtools** package. Then use **DEplot1** to plot the direction field. See Figure 50.

```
> with(DEtools):
> plt1 := DEplot1(x*(10 - x)/10,[t,x],t=0..5,x=-10..20,
> arrows = LINE):";
```

Note the line segments have positive, negative, or zero slope depending on where x is located. If $\phi(t)$ is a solution of the equation and $0 < \phi(t) < 10$, then $\phi'(t) > 0$. Thus in this range $\phi(t)$ is increasing. On the other hand if $\phi(t) > 10$ then $\phi'(t) < 0$ and $\phi(t)$ is decreasing. Is the behavior of these solutions eseentially the same as those of the previous example? At first glance you might feel that there is not much difference, but observe that for $\phi(t) < 0$ the slope is negative in Figure 50, but it is positive for Figure 47. The behavior is more complicated in this example.

A plot of the solution curves satisfying the initial conditions:

$$x(0) = 0, x(0) = 2, x(0) = 8, x(0) = 10, x(0) = 12, \text{ and } x(0) = 18$$

is given in Figure 51.

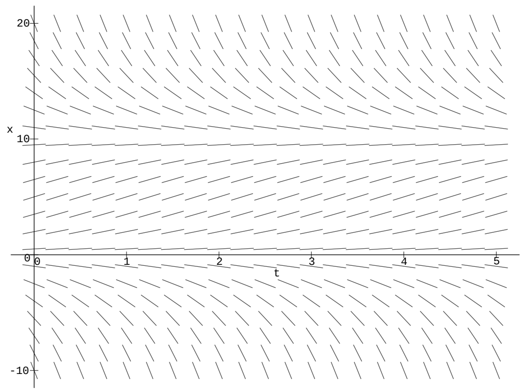

Figure 50: Direction field for $x' = \frac{x(10-x)}{10}$

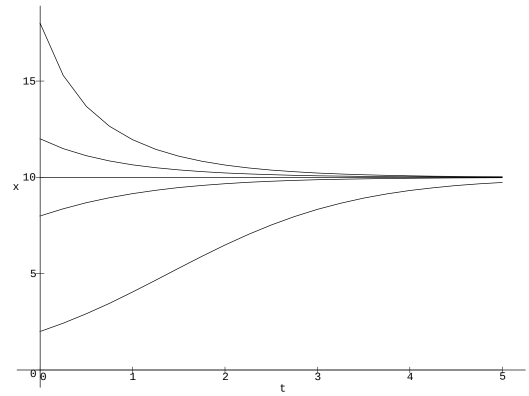

Figure 51: Five solution curves for $x' = \frac{x(10-x)}{10}$

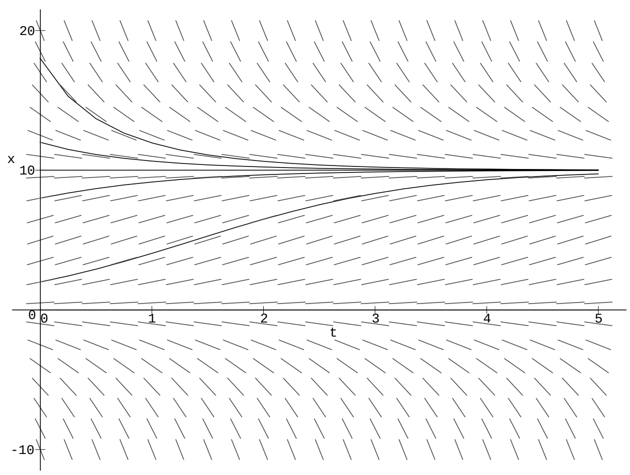

Figure 52: Solution curves and direction field

```
> plt2 := DEplot1(x*(10-x)/10,[t,x],t=0..5,{[0,0],[0,2],

> [0,8],[0,10],[0,12],[0,18]},x=-10..20,arrows=NONE): ";
```

Notice one difference with this example and the previous one is that the solution satisfying $x(0) = 0,$ is the constant function $\phi(t) = 0,$ and in the previous example all solutions tend to 2 as t tends to ∞. The union of the last two plots is given in Figure 52.

```
> plots[display]({plt1,plt2});
```

The preceeding two examples do not contain t explicitly. The next example does.

Example 9.2.3 Use **DEplot1** to analyze the time dependent system. Also duplicate the plots by using **dsolve** and the **plot** command.

Solution: Starting as with the previous examples, we call up the **DEtools** package and invoke **DEplot1** to create Figure 53.

```
> with(DEtools):
```

```
> plt1 := DEplot1(t+x,[t,x],t=-4..4,x=-4..4,arrows = LINE): ";
```

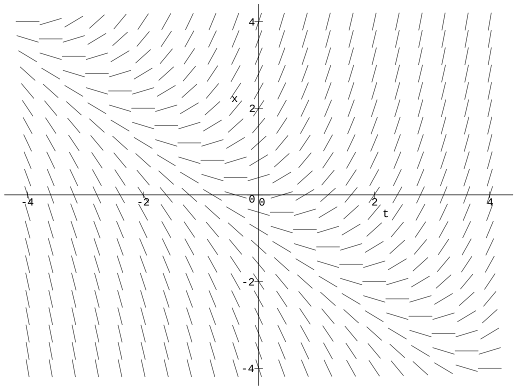

Figure 53: Direction field for $x' = t - x$

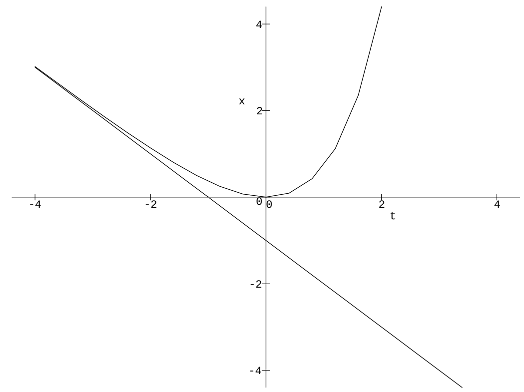

Figure 54: Two solution curves for $x' = t - x$

A plot of solution curves satisfying the initial conditions $x(0) = 0$ and $x(-1) = 0$ is shown in Figure 54.

```
> plt2 := DEplot1(t+x,[t,x],t=-4..4,{[0,0],[-1,0]},

> x=-4..4,arrows=NONE): ";
```

It turns out that we can solve this equation with **dsolve** and then plot the curve over any scale that is available to the **plot** command.

Define the equation in a Maple V session.

```
> deq := diff(x(t),t)=t+x(t);
```

$$deq := \frac{d}{dt}x(t) = t + x(t)$$

Now assign the initial conditions and use **dsolve**.

```
> init1 := x(0)=0;
                    init1 := x(0) = 0
> sol1 := dsolve({deq,init1},x(t));
```

$$sol1 := x(t) = -t - 1 + e^t$$

In order to use **plot** it is necessary assign the right-hand side to an expression.

```
> x1 := rhs(sol1);
```

$$x1 := -t - 1 + e^t$$

Now plots are in Figure 55.

```
> plt2 := plot(x1,t=-4..4,x=-4..4):";
```

We now use the other initial condition and obtain Figure 56.

```
> init2 := x(-1)=0;
                    init2 := x(-1) = 0

> sol2 := dsolve({deq,init2},x(t));
```

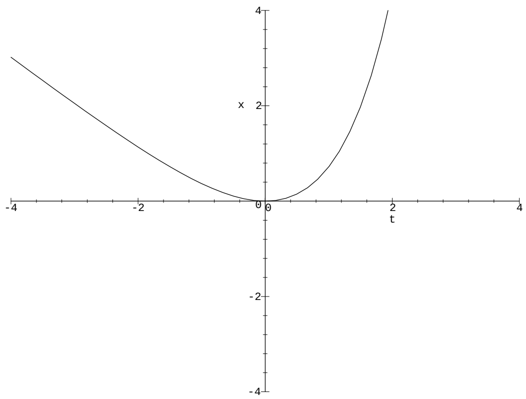

Figure 55: A solution curve for $x' = t - x$

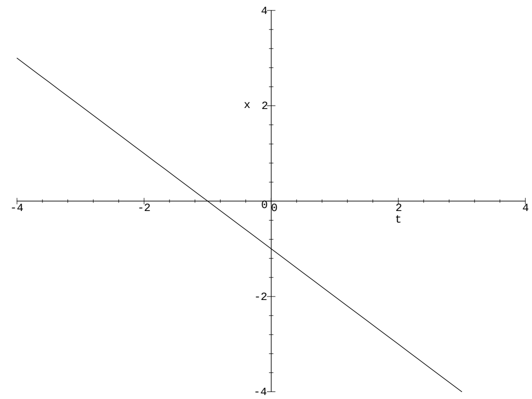

Figure 56: Another solution curve for $x' = t - x$

```
                    sol2 := x(t) = - t - 1
```

```
> x2 := rhs(");
```

```
                         x2 := - t - 1
```

```
> plt3 := plot(x2,t=-4..4,x=-4..4):";
```

The direction field together with these last two curves is plotted in Figure 57.

```
> plots[display]({plt1,plt2,plt3});
```

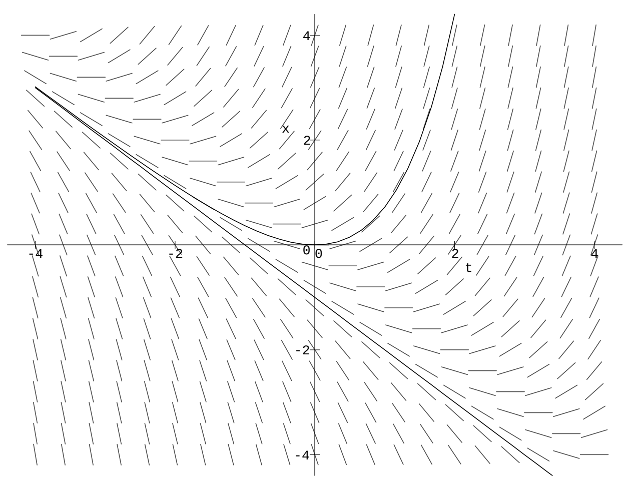

Figure 57: Solution curves and direction field

Note that Figure 57 can be created in one step with **DEplot1**.

Exercises 9.2 Using Maple V procedures plot the direction fields for the given differential equations in the indicated region of the (t, x)-plane . Include graphs of solution curves satisfying the indicated initial conditions.

1. $x' = -(t\ \sin t)\ \cos(x)$, where $-2\pi \le t \le 2\pi$, $-2\pi \le x \le 2\pi$. Initial conditions are $x(-2) = 2$, and $x(-1) = -3$.

2. $x' = \sin t\ \cos t$, where $-2\pi \le t \le 2\pi$, $-2\pi \le x \le 2\pi$. Initial conditions are $x(-2) = 2$, $x(-2) = 3$, and $x(0) = \pi$.

3. $x' = \cos(t - x)$, where $-2\pi \le t \le 2\pi$, and $-2\pi \le x \le 2\pi$. Initial conditions are $x(0) = 1$, $x(0) = 0$, and $x(0) = \pi$.

4. $x' = t^2 - x^2$, where $-4 \le t \le 4$, and $-4 \le t \le 4$. Initial conditions are $x(0) = 0$, $x(-1) = -3$, and $x(1) = 2$.

9.3 Euler's Method

Numerical approximations of derivatives and integrals that were based on their definitions were presented in previous chapters prior to introducing techniques and shortcuts that can be calculating them in "nice" cases. In that spirit, this section shows a method, (Euler's Method), for approximating the solutions of initial value problems for differential equations. The idea behind Euler's Method is simple. Suppose that you wish to find the solution of

$$x' = f(t, x)$$

which satisfies $x(t_0) = x_0$. Assume that $f(t, x)$ varies continuously with its variables. Choose a small interval of time, say h. Then since $f(t, x)$ is continuous we can hope that $f(t, x)$ is well approximated by $f(t, x_0)$, for $|t - t_0|$ and $|x - x_0|$ small. One can then find the solution of the constant differential equation

$$x' = f(t_0, x_0)$$

which satisfies $x(t_0) = x_0$. Integrating both sides gives the solution

$$\phi(t) = x_0 + f(t_0, x_0)\ (t - t_0).$$

The solution at time $t_1 = t_0 + h$ is

$$\phi(t_0 + h) = x_0 + f(t_0, x_0)\ h$$

and so write

$$x_1 = x_0 + f(t_0, x_0)\ h.$$

Do the same thing again starting at the point (t_1, x_1) and obtain the approximate solution for the interval $[t_1, t_1 + h]$ to obtain

$$t_2 = t_1 + h, \quad x_2 = x_1 + f(t_1, x_1)h.$$

After k steps it follows that

$$t_{k+1} = t_k + h, \quad x_{k+1} = x_k + f(t_k, x_k)h.$$

This means that the approximate value of the solution to the initifal value problem at time $t = t_k$ is $x_k + f(t_k, x_k)h$. An illustration of this will now be given by approximating the solution of

$$x' = t - x,$$

which satisfies $x(0) = 1$ using five iterations with $h = 0.1$

```
> f := (t,x) -> t - x;
```

$$f := (t, x) \mapsto t - x$$

```
> t[0] := 0; h:=0.1; x[0] := 1;
```

$$t_0 := 0$$

$$h := 0.1$$

$$x_0 := 1$$

```
> t[1] := t[0]+h; x[1] := evalf(x[0] + f(t[0],x[0])*h);
```

$$t[1] := .1$$

$$x_1 := .9$$

```
> t[2] := t[1]+h; x[2] := evalf(x[1] + f(t[1],x[1])*h);
```

$$t_2 := .2$$

$$x_2 := .82$$

```
> t[3] := t[2]+h; x[3] := evalf(x[2]+f(t[2],x[2])*h);
```

$$t_3 := .3$$

$$x_3 := .758$$

```
> t[4] := t[3]+h; x[4] := evalf(x[3]+f(t[3],x[3])*h);
```

$$t_4 := .4$$

$$x_4 := .7122$$

```
> t[5] := t[4]+h; x[5] := evalf(x[4]+f(t[4],x[4])*h);
```

$$t_5 := .5$$

$$x_5 := .68098$$

Thus the value of the solution to the initial value problem of the differential equation at times 0, 0.1, 0.2, 0.3, 0.4, 0.5 are 1, 0.9, 0.82, 0.758, 0.7122, 0.68098. In order to plot these points we create a list.

```
> L := [seq([t[i],x[i]],i=0..5)];

  L := [[0, 1], [.1, .9], [.2, .82], [.3, .758], [.4, .7122], [.5, .68098]]
```

We can now make a plot of the approximate solution Figure 58.

```
> plot(L);
```

You can verify by direct substitution that the exact solution to this problem is

$$\phi(t) = 1 - 2\,e^{-t}.$$

The error in the approximation at point t_i is

$$\text{ERROR} = \text{approximation at } t_i \text{ - } \phi(t_i)$$

This can be computed at each of the points 0, 0.1, 0.2, 0.3, 0.4, 0.5.

```
> error := seq(evalf(L[i][2]-(-1+2*exp(-(i-1)/10))),i=1..5);

          error := 0, .090325164, .182538494, .276363559, .371559908
```

You can compare the approximate and exact solution graphically as in Figure 59. For this problem the approximate solution is the higher one.

```
> plot({L,rhs(sol)},t=0..0.5);
```

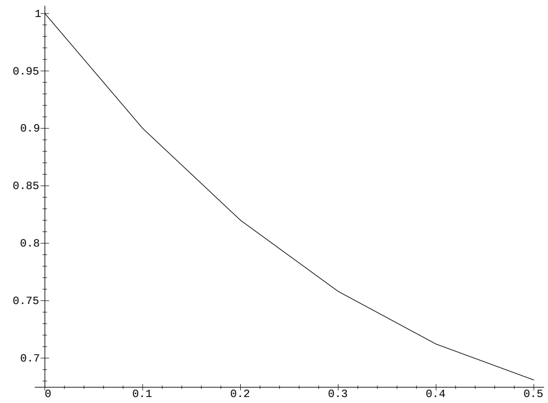

Figure 58: Euler's Method solution for $x' = t - x$

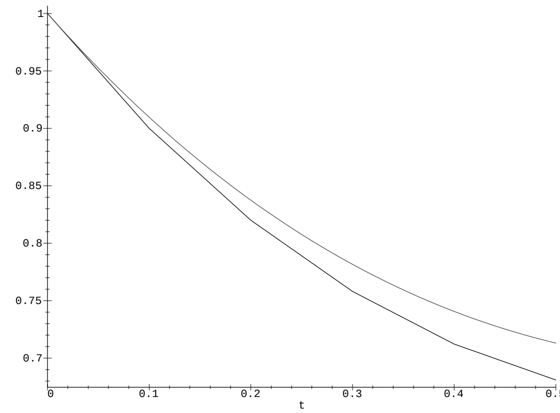

Figure 59: Approximate and exact solutions

When making a large number of iterations it is probably better to use a loop instead of typing in all of the repetitions. The following maple V segment shows how to make 50 iterations with $h = 0.01$.

```
> t[0] := 0: h:=0.01: x[0] := 1:

> i := 'i':


> for i from 0 to 49 do
> x[i+1] := x[i] + f(t[i],x[i])*h;
> t[i+1] := t[i]+h;


> od:


> x[50];

                              .7100121342
```

The new approximate value for $t = 0.5$ is $x[50]$ in this case and is .7100121342 which is a better approximation than the one with $h = 0.1$.

Exercises 9.3 Use Euler's Method with step size equal to $h = 0.1$ to determine an approximate value of the solution at $t = 1$ for each of the initial value problems below. Repeat these calculations with $h = 0.05$, and $h = 0.01$ and compare the result with the exact value of $x(1)$. You may use **dsolve** to obtain the exact solution. Graph the result along with the direction field in each case.

1. $x' = x,\ x(0) = 1$
2. $x' = t + x,\ x(0) = 1$
3. $x' = t - x,\ x(0) = 2$
4. $x' = 3x - 4e^{-t}, x(0) = 1$
5. $x' = x(10 - x), x(0) = 2$

9.4 Separation of Variables

In the preceding sections you have seen how to use the direction field defined by a differential equation to gain geometric insight into how the solutions behave, and how to use Euler's Method to numerically approximate the solutions. In this section a method for finding the exact solution, in cases when the differential equation is given in the following special form:

$$x' = T(t)X(x).$$

Definition
A first-order differential equation is *separable* if it can be written in the form:

$$x' = T(t)X(x).$$

As an example consider the following differential equation:

$$x' = (2t+1)x.$$

This equation is in separable form with $T(t) = 2t+1$, and $X(x) = x$. Suppose that $\phi(t)$ is a solution of $x' = (2t+1)x$, then $\phi(t)$ satisfies

$$\phi'(t) = (2t+1)\phi(t).$$

If $\phi(t) = 0$ then the right hand side of the equation is zero which means that the constant function with value zero

$$\phi(t) = 0$$

satisfies the equation. More generally, if $\phi(t) \neq 0$ one can divide both sides of the equation by $\phi(t)$ obtaining

$$\frac{\phi'(t)}{\phi(t)} = 2t+1.$$

Observe that both sides of this last equation can be integrated with respect to t

$$\int \frac{\phi'(t)}{\phi(t)}\,dt = \int (2t+1)\,dt.$$

Calculating the integral on each side we obtain

$$\ln\phi(t) = t^2 + t + C.$$

In order to solve for the solution $\phi(t)$ we apply the inverse function, exp, to both sides. Thus

$$\phi(t) = e^{t^2+t+C}.$$

This can be written as

$$\phi(t) = Ke^{t^2+t},$$

where $K = e^C$. The constant, K, can be evaluated by solving the last equation when $t = 0$, and a formula for ϕ is found:

$$\phi(t) = \phi(0)\,e^{t^2+t}.$$

Once you have found a candidate for a solution to a differential equation it is always a good idea to check to see if it really satisfies the equation. Upon differentiating the equation for ϕ, with the help of the chain rule, you arrive at

$$\phi'(t) = \phi(0)e^{t^2+t}\,(2t+1) = \phi(t)(2t+1) = (2t+1)\phi(t).$$

Thus $\phi(t)$ is a solution.

In general when an equation has the separable form you can obtain at least a implicit expression for a solution. Suppose then that a differential equation of the form

$$x' = T(t)\ X(x)$$

is given. How does one find the solution? In the example above it is assumed that $x = \phi(t)$ is a solution and then both sides are divided by $x = \phi(t)$. Since in that example $X(x) = x$, divide the general separable equation by $X(\phi(t))$. Then

$$\frac{\phi(t)'}{X(\phi(t))} = T(t).$$

Let $G(x)$ denote the antiderivative of the function $\frac{1}{X(x)}$ then if both sides of the last equation are integrated with respect to t one obtains

$$G(\phi(t)) = \int T(t)\ dt + C,$$

where the fact that

$$G(x) = \int \frac{dx}{X(x)},$$

implies

$$G(\phi(t)) = \int \frac{\phi'(t)\ dt}{X(\phi(t))}.$$

The equation

$$G(\phi(t)) = \int T(t)\ dt + C$$

defines the function

$$\phi(t)$$

implicitly. Since

$$G'(x) = \frac{1}{X(x)} \neq 0,$$

the function G has an inverse, G^{-1}, thus $\phi(t)$ is given by

$$\phi(t) = G^{-1}(\int T(t)\ dt + C).$$

Often one can't find an elementary formula for G^{-1}, but in any case the solution $x = \phi(t)$ is given implicitly by the relation:

$$G(x) - \int T(t)\ dt = C.$$

A function $F(t, x)$ like

$$F(t, x) = G(x) - \int T(t)\ dt$$

which is constant when a solution is substituted for x is called an *integral* for the differential equation. Thus for the differential equation

$$x' = (2t + 1)x$$

the function

$$F(t, x) = \ln x - e^{t^2+t}$$

is an integral for the differential under discussion.

Example 9.4.1 Use the method of separation of variables to find the solutions of the logistic type equation

$$x' = x(x - 1).$$

Solution: This equation is in the separable form with $T(t) = 1$, and $X(x) = x(x-1)$. The process starts by dividing the equation by $x(x-1)$. With Maple V the session begins as follows.

```
> deq := diff(x(t),t) = x(t)*(x(t)-1);
```

$$deq := \frac{d}{dt}x(t) = x(t)\,(x(t)-1)$$

```
> deqsep := deq/(x(t)*(x(t)-1));
```

$$deq := \frac{\frac{d}{dt}x(t)}{x(t)\,(x(t)-1)} = 1$$

Now the variables are separated with the function of x on the left-hand side and the function of t on the right-hand side. Each side can be integrated.

```
> intlhsdeqsep := int(lhs(deqsep),t);
```

$$intlhsdeqsep := -\ln(x(t)) + \ln(x(t)-1)$$

```
> intrhsdeqsep := int(rhs(deqsep),t)+C;
```

$$intrhsdeqsep := t + C$$

This means that

$$F(t,x) = -\ln(x) + \ln(x-1) - t$$

is an integral of the differential equation $x' = x(x-1)$. Now solve the equation for $x(t)$.

```
> phi := solve(intlhsdeqsep=intrhsdeqsep,x(t));
```

$$\phi := \frac{1}{\left(1 - e^{t+C}\right)}$$

Sometimes it is desirable to make ϕ a function.

```
> phi := unapply(phi,t);
```

$$\phi := t \mapsto \frac{1}{1 - e^{t+C}}$$

Thus the general solution is

$$\phi(t) = \frac{1}{1 + e^{t+C}}.$$

The following steps represent a check to see if $\phi(t)$ really is a solution. The candidate for a solution is substituted into the differential equation

```
> eval(subs(x = phi,deq));
```

$$\frac{e^{t+C}}{\left(1 - e^{t+C}\right)^2} = \frac{\left(1 - e^{C}\right)^{-1} - 1}{1 - e^{t+C}}$$

Simplification illustrates the validity of the solution.

```
> simplify(");
```

$$\frac{e^{t+C}}{\left(-1 + e^{t+C}\right)^2} = \frac{e^{t+C}}{\left(-1 + e^{t+C}\right)^2}$$

Since both sides are equal we have checked the correctness of the solution.

In this section the method of separation of variables is being emphasized. Now an illustration of how to solve this equation using **dsolve** will be presented.

```
> dsolve(deq,x(t));
```

$$\frac{1}{(x(t))} = 1 + e^t\_C1$$

Note that **dsolve** did not solve for the solution explicitly, but it is easy to obtain the explicit solution.

```
> solve(",x(t));
```

$$(x(t)) = \frac{1}{1 + e^t\_C1}$$

This says that the general solution is

$$x(t) = \frac{1}{1 + e^t\_C1}.$$

You should be able to prove that this is equivalent to the one obtained above.

When you can find the solution explicitly, you can also use the Maple V option **explicit** to ask **dsolve** to return that solution.

```
> dsolve(deq,x(t),explicit);
```

$$x(t) = -\frac{1}{(-1 - e^t\_C1)}$$

When a differential equation is separable you have a chance to find an exact solution. Sometimes it is not easy, (or even possible), to evaluate the integral

$$\int \frac{dx}{X(x)}.$$

That can be an obstruction to finding an exact solution. At other times you can not obtain the solution explicitly. Nevertheless the method of separation of variables appears in applications often enough to be studied. The next example is an illustration of a problem in which Maple V's ability to evaluate integrals helps a lot.

Example 9.4.2 Find the solution of

$$x' = \frac{t^2}{e^x \cos(x)\sqrt{9 - t^2}},$$

which satisfies the initial condition $x(0) = 0$. Plot the graph of the solution.

Solution: This equation is separable with $T(t) = \frac{t^2}{\sqrt{9-t^2}}$, and $X(x) = e^x \cos x$. Start out just like in the last problem.

```
> deq := diff(x(t),t)= (t^2)/(exp(x(t))*cos(x(t))*\sqrt(9-t^2));
```

$$deq := \frac{d}{dt}x(t) = \frac{t^2}{e^{x(t)} \cos(x(t))\sqrt{9 - t^2}}$$

Now separate variables

```
> deqsep := deq * (exp(x(t))*cos(x(t)));
```

$$deqsep := e^{x(t)} \cos(x(t))\frac{d}{dt}x(t) = \frac{t^2}{\sqrt{9 - t^2}}$$

and integrate both sides.

```
> intlhsdeqsep := int(lhs(deqsep),t);
```

$$intlhsdeqsep := \frac{e^{x(t)} \cos(x(t))}{2} + \frac{e^{x(t)} \sin(x(t))}{2}$$

```
> intrhsdeqsep := int(rhs(deqsep),t)+C;
```

$$intrhsdeqsep := -\frac{t\sqrt{9 - t^2}}{2} + \frac{9 \arcsin(\frac{t}{3})}{2} + C$$

The integral follows by equating the last two results.

```
> integral := intlhsdeqsep = intrhsdeqsep;
```

$$integral := \frac{e^{x(t)}\cos(x(t))}{2} + \frac{e^{x(t)}\sin(x(t))}{2} = -\frac{t\sqrt{9-t^2}}{2} + \frac{9\arcsin(\frac{t}{3})}{2} + C$$

An explicit solution, in this case, seems impossible, so leave the solution in implicit form. Now solve for C by using the initial condition.

```
> inits  := eval(solve(subs({t=0,x=0},integral),C));
```

$$inits := 1/2$$

Substitute this value of C into the integral.

```
> initintegral := subs(C=1/2,integral);
```

$$initintegral := \frac{e^{x(t)}\cos(x(t))}{2} + \frac{e^{x(t)}\sin(x(t))}{2} = -\frac{t\sqrt{9-t^2}}{2} + \frac{9\arcsin(\frac{t}{3})}{2} + 1/2$$

One can now plot the solution using **implicitplot**. See Figure 60.

```
> plots[implicitplot](initintegral,t=-3..3,x=-2*Pi..2*Pi);
```

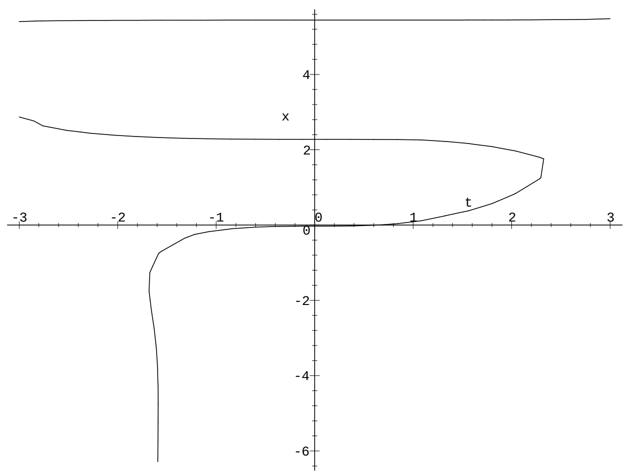

Figure 60: Solution Curve

Exercises 9.4 Use the method of separation of variables to solve the following initial value problems. Whenever possible find explicit solutions. In all cases plot the solution.

1. $x' = t(1+x^2),\ x(0) = 1$
2. $x' = \sin t \sin x,\ x(0) = \frac{\pi}{4}$
3. $x' = 0.005x(500-x),\ x(0) = 20$
4. $x' = \frac{1}{x \ln x},\ x(0) = 1$
5. $x' = \frac{t(1+x^2)}{(1+t^2)},\ x(0) = 1$

9.5 Models of Growth and Decay: First Order Rates

Much of the work. that scientists and engineers do, involves the modeling process. In previous sections we have made reference to physical laws such as, for example, Newton's law of Cooling, or Hooke's Law. These are examples of famous mathematical models and have been accepted by elements of the scientific community over such a long time that they are called "laws". Much of the routine daily activity of practicing scientists involves developing some kind of model. The process might start with some kind of " real-world" problem, which comes up in the scientist's discipline, that may only be vaguely understood, but for which there are good reasons to have more understanding. The goal to be achieved must be articulated. The process might go like this. In the first step one determines components affecting the behavior of the problem under consideration, and isolates those mechanisms that are important in terms of the overall goals. The problem is then cut down to a manageble size. The next step is to determine constraints and scientific laws that apply to the specialized problem. Most mathematical models have the following elements: (1) a mathematical or logical structure, (algebraic formulas, differential equations, *etc.*), (2) definitions of the variables involved, and (3) the distinquishing features within the mathematical structure of all laws and constraints that are relevent to the problem. Once a model has been obtained, it can be analyzed through its own internal mathematical structure so that the behavior of its variables can be predicted. Thus the process that starts with a "real-world" problem leads to a mathematical problem, which can be analyzed to obtain a prediction about the original problem. The predictions made as a result of the model may or may not agree with experimental results, or might suggest new laboratory experiments. If the model does not give realistic predictions then one must return to the model, determine which assumptions made during the process have led to these incorrect predictions, and then make revisions to the model accordingly.

Radioactive Decay

As a radioactive material loses some of its mass as radiation energy, the remainder of the material reforms to create a new substance. This process is called *radioactive decay*. For example, as radioactive carbon-14 decays it forms nitrogen. The ultimate result of the decay of radium is lead. Experiments have shown that at any given time, the rate at which a radioactive element decays is proportional to the mass of the element that is present. Let $x(t)$ denote the mass of a radioactive substance at time t. Its rate of decay has the form

$$x' = -kx.$$

If x_0 is the mass at time $t = 0$, then

$$x(t) = x_0 e^{-kt}.$$

The *half-life* of a radioactive substance is the time required for half of the substance to have decay. It is related to k by solving for T_{half} in the equation

$$x_0 e^{-kT_{half}} = \frac{x_0}{2}.$$

Thus

$$T_{half} = \frac{\ln(2)}{k}.$$

Example 9.5.1 A living substance is assumed to have the same proportion of carbon-14 as the atmosphere has and stops absorbing carbon when it dies. This means that the proportion of carbon-14 in, say, a plant that was once alive can be used as an indicator of how long ago the plant died. The half-life of carbon-14 is 5700 years. Suppose that a sample has 90% of the carbon-14 that it originally had. Find the age of the sample.

Solution: Let $x(t)$ denote the amount per gram of carbon-14 per gram of carbon in sample at time t. Then $x(t) = x_0 e^{-kt}$. Since the half-life of carbon-14 is 5700 years, we can determine k from the formula

$$k = \frac{\ln(2)}{5700}.$$

The time T in years back at which the sample died satisfies

$$x_0 e^{-kT} = x_0 \cdot (9/10),$$

or

$$e^{-kT} = 9/10.$$

Solving for T gives

$$T = -\frac{5700 \ln(0.9)}{\ln(2)}.$$

```
> T = evalf(-5700*ln(0.9)/(ln(2)));
```

```
T = 866.4176331
```

It can be concluded that the sample has been dead for at least 866 years.

Drag Near the Earth's Surface

A body of low density and rough exterior (*e.g.*, a feather, or a snowflake), moving near the earth's surface has a resistive force due to air which is propotional to the velocity, v, but acts opposite to the motion. Thus if such a body has mass m is released at height x_0 with initial velocity v_0 in the vertical direction has a force due to gravity and resistance equal to

$$F = -mg - kv,$$

where g is the acceleration due to gravity, and $k > 0$ is a constant of proportionality. Using Newton's Second Law it can be seen that $v(t)$ satisfies the initial value problem

$$mv' = mg - kv, \quad v(0) = v_0.$$

Example 9.5.2 Suppose that the velocity $v(t)$, of a body of low density satisfies the initial value problem

$$v' = -100 - 0.04 \cdot v, \, v(0) = 0.$$

Find the limiting velocity.

Solution: In this problem the differential equation is given so all that has to be done is to solve it.

```
> deq := diff(v(t),t) = -100- 0.4*v(t);
```

$$deq := \frac{d}{dt} v(t) = -100 - 0.4 v(t)$$

```
> dsolve({deq,v(0)=0},v(t));
```

```
Error, (in factor/factor) floats not handled
```

The error here occurs because when one is trying to find the exact solution using **dsolve** the differential equation can not use floating point numbers. There are several ways to remedy this in this case. One way be to rewrite 0.4 as $4/10 = 2/5$.

```
> deq := diff(v(t),t) = -100- (2/5)*v(t);
```

$$deq := \frac{d}{dt} v(t) = -100 - \frac{2\, v(t)}{5}$$

The **dsolve** procedure works in this case and the exact solution can be found as follows.

```
> dsolve({deq,v(0)=0},v(t));
```

$$v(t) = -250 + 250\, e^{-\frac{2t}{5}}$$

The limiting velocity is -250 units of velocity. The significance of the negative sign is that it means that the object is falling.

In other situations, such as, if the falling object is dense (*e.g.*, a raindrop, baseball or bullet), and moves near the earth's surface, the resistive force of the air might be proportional to the square of the speed and acts opposite to the direction of the motion. Therefore the equation of motion has the form

$$mv' = -mg \pm kv^2,$$

where $k > 0$ is the drag coefficient; the upper sign (+) is chosen if the body is falling, and the lower sign (-) if the body is rising. The equation can also be written

$$mv' = -mg - v|v|.$$

Example 9.5.3 Suppose a smooth dense object falls with velocity which satisfies the differential equation:

$$v' = -1 - v|v|$$

Find the limiting velocity in the case of an initial velocity of $v(0) = 0$, and plot several solution curves with various initial velocities.

Solution: In this example **dsolve** with the **numeric** option will be used to solve the given initial value problem.

```
> deq := diff(v(t),t)=-1-v(t)* abs(v(t));
```

$$deq := \frac{d}{dt}v(t) = -1 - v(t)\,|v(t)|$$

```
> sol := dsolve({deq,v(0)=0},v(t),type=numeric);

sol := proc(rkf45_x) ... end
```

We can estimate the limiting velocity.

```
> seq(sol(i)[2],i=3..5);

            v(t) = -.9950558671716422, v(t) = -.9993294546617530,

                v(t) = -.9999092277555853
```

It appears to be $v = -1$. The velocity tends to approach the constant solution $v(t) = -1$. Next use **DEplot1** to plot several solution curves. See Figure 61

```
> with(DEtools):

> DEplot1(deq,v(t),t=0..5,{[0,-2],[0,-1],[0,0],[0,1],[0,2],[0,3]},

> arrows = LINE,v=-2..3);
```

Population Models

Let $P(t)$ denote population at time t of a species. In reality the values of $P(t)$ are integers, and they change by integral amounts with time. However, for large populations a change of one or two is "infinitessimal" relative to the total, and we may think of $P(t)$ as a continuous or even a smooth function, and thus we hypothesize the existence of the rate of change of the population, $P'(t)$. This leads to differential equations. In general the following relation is assumed to hold:

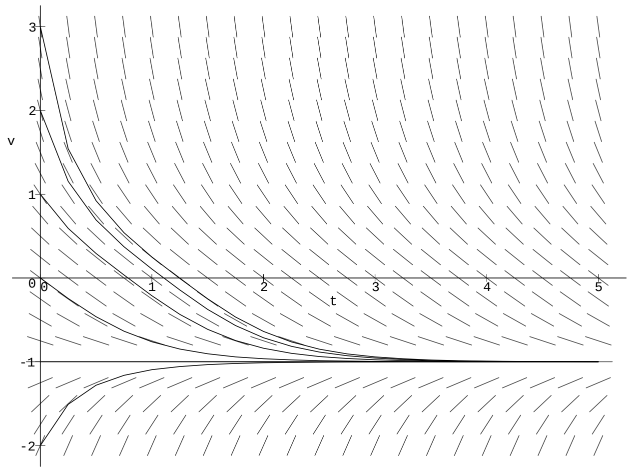

Figure 61: Solution curves for velocity of smooth, dense falling object

Rate of Change = rate in - rate out

Example 9.5.4 Suppose that a population is isolated in the sense that there are no outside influences (*i.e.*, there is no immigration or or emigration) and that the only change in the population is due to births and deaths. In this case the *rate in* is the birth rate and is assumed to be linearly proportional to the the size of the population

rate in = b· P(t)

The *rate out* is the death rate and is also assumed to be linearly proportional to to the size of the population.

rate out = d · P(t)

Then the size of the population $P(t)$, with initial size, $P(0)$, satisfies the initial value problem:

$$P' = (b-d)P,$$

$$P(0) = P_0.$$

The solution of this initial value problem can be obtained by finding the explicit solution using the method of separation of variables to be

$$P(t) = P_0 e^{(b-d)t}.$$

If the birth rate exceeds the death rate then the population grows exponentially, but if the death rate is larger than the population "dies out" exponentially.

Example 9.5.5 Suppose that a population initially has a birth rate constant of proportionality $b = 0.06$, births/year and a death rate of $d = 0.04$ deaths/year. After 15 years of steady growth, assume that the population stops reproducing, *i.e*, $b = 0$. Find how long after the population stops reproducing that it takes the population

1. to return to its original level
2. to reach 50% of its original level
3. to reach 30% of its population at the time b became zero.

Solution: In this case the differential equation changes with time. For the first fifteen years we have

$$P' = 0.02 \cdot P,$$

which has solution

$$P(t) = P_0 e^{0.02t}.$$

After 15 years the differential equation is

$$P' = -0.4 \cdot P.$$

We can define the solution piecewise by

$$P(t) = P_0 e^{0.02t},$$

for $0 \le t \le 15$, and

$$P(t) = P_{15} e^{-0.4(t-15)},$$

for $15 < t$, where $P_{15} = P_0 e^{.30}$. You need to verify this. Think of solving the initial value problem

$$P' = -0.04P, \quad P(15) = P_0 e^{0.02 \cdot 15}.$$

Using the Heaviside Function, H, you can define the solution

$$P(t) = P_0 e^{0.02t}(H(t) - H(15) + P_0 e^{.30} e^{-0.04(t-15)} H(t-15).$$

A plot of the solution curve along with the horizontal lines corresponding to the original population, 50% of the original population, and 30% of the population at the time b becomes zero is shown in Figure 62 where we have set $P_0 = 1$ in order to plot the graph.

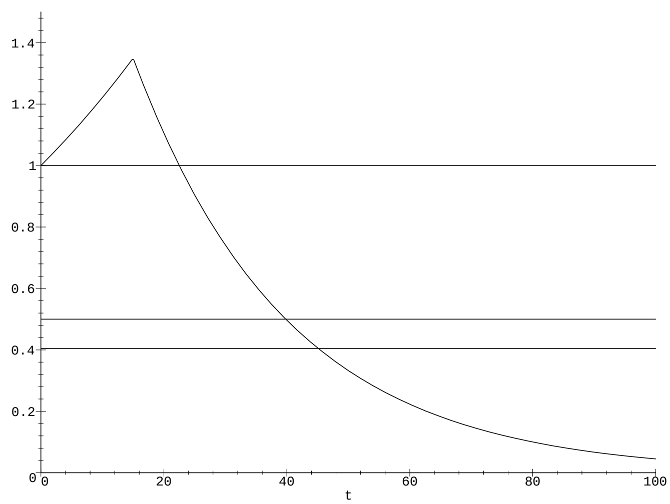

Figure 62: Population in which reproduction stops

In our Maple V session we use the notation *Pbefore* to denote the population before reproduction halts and *Pafter* to denote the population after this event.

```
> Pbefore := (P0,t) -> P0*exp(0.02*t);
```

$$(P0, t) \mapsto P0\, e^{0.02t}$$

The population at the time reproduction seizes is found as follows.

```
> P15 := Pbefore(P0,15);
```

$$P15 := .349858808\,P0$$

Now compute *Pafter*.

```
> Pafter := (P0,t) -> Pbefore(P0,15)*exp(-0.04*(t-15));
```

$$Pafter := (P0, t) \mapsto 1.349858808\,P0\,e^{-0.04t+0.60}$$

Use the Heaviside Function and plot Figure 62.

```
> alias(H=Heaviside);
                                I, H
> P := Pbefore(1,t)*(H(t)-H(t-15)) + Pafter(1,t)*H(t-15);
```

$$P := e^{0.02t}\,(Heaviside(t) - Heaviside(t-15)) + 1.349858808e^{-0.04t+0.60}\,Heaviside(t-15)$$

```
> plt1 := plot(P,t=0..100,0..1.5): ";

> plt2 := plot(1,t=0..100):

> plt3 := plot(1/2,t=0..100):

> plt4 := plot(subs(P0=1,0.30*P15),t=0..100):";

> plots[display]({plt1,plt2,plt3,plt4});
```

We now solve the problem numerically.

```
> solve(Pafter(P0,t) = P0,t);

                              22.50000001

> solve(Pafter(P0,t)=P0/2,t);

                              39.82867952

> solve(Pafter(P0,t)=0.30*P15,t);

                              45.09932010
```

Using the preceding Maple V segment one can now answer the questions posed in the example. You may conclude that it takes

$$22.5 - 15 = 7.5$$

years for the population to return to its original level. The population returns to its original population in half of the time that the population grew to its maximum value. The population reaches 50% of the original population in

$$39.82867952 - 145 = 24.82867952$$

years. Finally, the population decreases to 30% of its population at the time *b* became zero in

$$45.09932010 - 15 = 30.09932010$$

years.

The "explosive" growth that arises when a population satisfies a linear growth rate is not always realistic, since the exponential increase will soon outstrip the resources that are necessary to support the population. One way to model restricted population growth is to assume that the rate coeffient is variable rather than constant. Assume that this coefficient is linear, the next simplest after the constant case, then the population can be assumed to satisfy the *logistic equation*

$$P' = r(L - P)P.$$

The factor $L - P$ is called the *limiting factor*. This problem can be solved by the method of separation of variables. There have been several versions of this type of equation analyzed in previous sections.

Example 9.5.6 It is known that the resources of a certain region can sustain at most 250 wolves. There are presently 25 wolves in the region. Assume that the population of wolves grows at a logistic rate and the constant of proportionalty is $r = 0.001$ wolves/year.

1. Determine the population of wolves $P(t)$ as an explicit function of time.
2. Plot the graph of $P(t)$.
3. What values of P and t make sense in the problem situation?
4. When will the wolf population reach 100?
5. When will the population essentially reach its limit?

Solution:

1. The population of the wolves must satisfy the equation

$$P' = .001P(250 - P),$$

with $P_0 = 25$.

```
> P := 'P';
```

```
> deq := diff(P(t),t) = 1/1000*P(t)*(250-P(t));
```

$$deq := \frac{d}{dt}P(t) = \frac{P(t)\,(250 - P(t))}{1000}$$

Using **dsolve** with the **explicit** option, obtain the explicit solution and denote it by PW.

```
> PW := dsolve({deq,P(0)= 25},P(t),explicit);
```

$$PW := P(t) = -\frac{250}{-1 - 9\,e^{-\frac{t}{4}}}$$

The expression PW is converted to a function using **unapply**.

```
> PW := unapply(rhs(PW),t);
```

$$PW := t \mapsto -\frac{250}{-1 - 9\,e^{-\frac{t}{4}}}$$

2. Now plot the graph of $P(t)$. See Figure 63.

```
> plot(PW(t),t=-10..50,-50..300);
```

3. Since the population can never be negative, nor exceed 250, and since time is measured from the present, it follows that:

$$0 \le t, \quad 0 \le PW \le 250.$$

4. Use **solve** to predict when the population will reach 100.

```
> solve(PW(t)=100,t);
```

```
4 ln(6)
```

```
> evalf(");
```

```
7.167037876
```

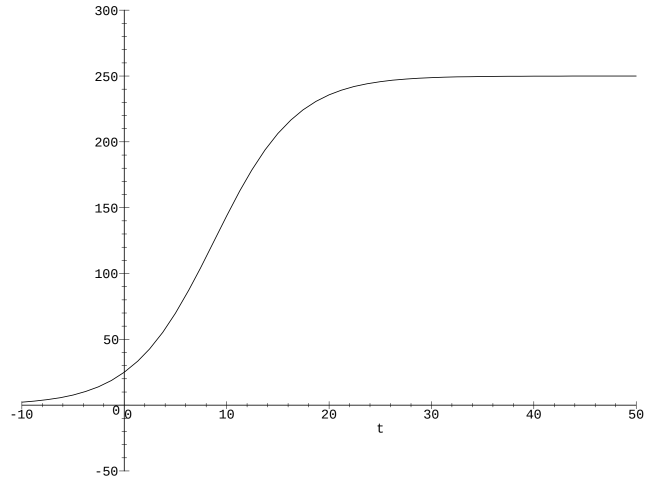

Figure 63: Graph for P(t)

This indicates that there will be 100 wolves in about 7.167037876 years.

5. The wolf population will reach 249 (one less than the limit) in

```
> solve(PW(t)=249,t);
                                4 ln(2241)

> evalf(");
                                30.85870990
```

Therefore the population can be considered to reach its limiting population in around 31 years. See Figure 64.

```
> plot(PW(t),t=0..40,0..250);
```

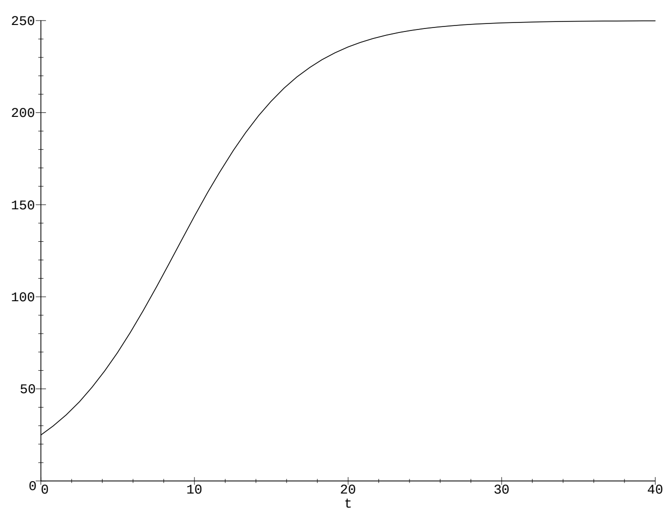

Figure 64: Graph for wolf population

Equilibrium Solutions

Most of the differential equations that have been used as models have the form

$$x' = f(x).$$

For example in the example on Newton's Law of Cooling the differential equation had the form:

$$T' = -k(T - T_s).$$

In the equation for radioactive decay the differential had the form:

$$x' = -kx.$$

The equation of population growth has the form

$$P' = (b - d)P.$$

The equations for logistic population growth of wolves in the last example was

$$P' = -.001P(250 - P).$$

In each case these equations have points where the right hand side of the equation vanishes. These are points in which the derivative of the solution is zero. Since the solution curve through such a point has zero slope the solution must be constant, *i.e.,* is in equilibrium. Such points are called *equilibrium points,* and the constant solution is called an *equilibrium solution*. How do solutions of a differential equation behave near an equilibrium point? In the model involving Newton's law of Cooling the equilibrium point is $T = T_s$, the temperature of the surrounding area. All solutions tend to this temperature as $t \to \infty$. One says that an equilibrium solution that has this property is *stable.* In the population qrowth equation the equilibrium occurs at $P = 0$, the general is

$$P(t) = P_0 e^{(b-d)t}.$$

If the birth rate is larger then the death rate then the growth "explodes" as t increases, but note that as $t \to -\infty$ the solution tends to the equilibrium solution. When this happens it is said that the equilibrium solution is *unstable.* On the other hand, if the death rate exceeds the birth rate all solutions tend to the equilibrium solution $P = 0$, *i.e.,* the population dies out, *i.e.,* the equilibrium solution is stable. Finally, in the logistic equation for wolves there are two equilibrium points: $P = 0$ and $P = 250$. The solution of the initial value problem used for this equation was

$$P(t) = -\frac{250}{-1 - 9e^{-\frac{t}{4}}}.$$

This solution tends to the equilibrium solution $P = 250$ as $t \to \infty$ and the solution goes to the other equilibrium position as $t \to -\infty$. One can show that any solution with initial value satisfying $0 < P(0) < 250$ has these properties. Thus in the case of the model involving the wolves the equilibrium solution $P = 250$ is stable, and the equilibrium solution $P = 0$ is unstable. Consequently, if the wolves have been living in the region for a number of years you would expect to find around 250 wolves living there.

In the qualitative study of mathematical models that use differential equations the equilibrium solutions are important in that they are the solutions to which the system seems to tend to or to tend away from with increasing time.

Let $x' = f(x)$ be a first order differential which does not involve the independent variable t explicitly. Let $f(c) = 0$. We say that $x = c$ is an *equilibrium point*.

- An *equilibrium solution* is a constant solution, $\phi(t) = c$, where $f(c) = 0$.
- An equilibrium solution is *stable* if a small change in the initial conditions gives a solution which approaches the equilibrium point as $t \to \infty$.
- An equilibrium solution is *unstable* if a small change in initial conditions gives a solution curve that moves away from from the equilibrium point as $t \to \infty$.

Exercises 9.5

1. A certain radioactive substance has a half-life of 1740 years.

   (a) Write a differential equation describing the decay of a sample of this substance and plot a graph of a solution. From your graph of sample mass versus time, estimate the time required for the sample to decrease to 25

   (b) Compute this time from a solution formula for the sample size and compare with your result in (b).

2. At time $t = 0$ a parachutist who weighs 165 lbs opens the parachute at the height of 3000 ft when the velocity is 88 ft/sec. The force of air resistance is given by $60v(t)$ lbs/sec, where $v(t)$ the velocity of the parachutist at time t.

   (a) Write out a differential equation and initial value problem for $v(t)$, and another differential equation and initial value problem for $x(t)$, the height of the parachutist above the ground t.

   (b) Create a graph of v versus t, $0 \leq t \leq 1$ min. What value does $v(20)$ have?

   (c) When will the parachutist hit the ground?

   (d) What will the parachusist's velocity be just before hitting the ground?

3. An aquarium can support no more than 225 tropical fish of a certain species. Nine of these fish are placed into the aquarium. Assume that the rate of growth P' of the fish is directly proportional to the population P and the limiting factor $225 - P$ at any time t in weeks with proportionality constant $r = 0.00225$.

   (a) Determine the fish population $P(t)$ as an explicit function of time t.

   (b) Make a Maple V plot of $P(t)$.

   (c) What values of P and t make sense in the problem situation.

   (d) Make a direction field of the problem situation.

   (e) When will the fish population be 100? 150?

   (f) When will the fish population essentially reach the aquarium's capacity?

9.6 Systems of Differential Equations

In the previous sections of this chapter it was shown how to analyze a single differential equation. Sometimes an exact solution can be found explicitly, but in many cases one can only approximate the solutions numerically. Even in the latter case it is possible to determine many of the salient features of the solutions by studying their direction fields. Analyzing the behavior of solutions near equilibrium points and determining their stability, gives much insight into the long term qualitative behavior of solutions. You should now be aware, from the examples of mathematical models you have seen, that understanding the behavior of solutions to certain differential equations leads to predictions about solutions of "real-world" problems. In this section we study the behavior of solutions of systems of more than one differential equation.

Definition
Let $f(t, x, y)$ and $g(t, x, y)$ be functions of three variables t, x and y. A system of equations of the form

$$x' = f(t, x, y), \; y' = g(t, x, y)$$

is called a **first order system of differential equations**. The variables x, and y are the *dependent variables* and the variable t is called the *independent variable.* Differentiable functions $\phi(t)$ and ψ which are defined on some interval I and such that the equations

$$\phi'(t) = f(t, \phi(t), \psi(t)), \; \psi'(t) = g(t, \phi(t), \psi(t))$$

are satisfied for all $t \in I$ is called a **solution** of the system of differential equations.

For example, the functions $\phi(t) = \sin t,\ \psi(t) = \cos t$ provide a solution for the system of differential equations

$$x' = y,\ y' = -x,$$

as can be seen by direct substitution.

Example 9.6.1 Analyze the solutions of the system of differential equations

$$x' = -xy,\ y' = xy - y.$$

Solution: Maple V will be used to study the solutions of this system numerically and geometrically.

```
> deq := diff(x(t),t)=-x(t)*y(t),diff(y(t),t)=x(t)*y(t)-y(t);
```

$$deq := \frac{d}{dt}x(t) = -x(t)y(t),\ \frac{d}{dt}y(t) = x(t)y(t) - y(t)$$

As an illustration one can obtain a numerical solution of the system using **dsolve** with the **numeric** option satisfying the initial value problem

$$x(0) = 2.5, y(0) = 0.1.$$

```
> sol := dsolve({deq,x(0)=2.5,y(0)=0.1},{x(t),y(t)},type=numeric);

sol := proc(rkf45_x) ... end

> sol(0); sol(2); sol(4);

          [t = 0, x(t) = 2.500000000000000, y(t) = .1000000000000000]

          [t = 2, x(t) = 1.174952482169065, y(t) = .6699844908362961]

          [t = 4, x(t) = .3853721657880563, y(t) = .3447913533442888]
```

The above segment solves the initial value problem. The last maple V output shows that if $(\phi(t), \psi(t))$ denotes the solution then for values of $t = 2$ and $t = 4$ we have

$$\phi(2) = 1.174952482169065, \psi(2) = .6699844908362961$$

and

$$\phi(4) = .3853721657880563, \psi(4) = .3447913533442888.$$

This solution can be used to create plots, but rather, we will use **Plot2** from **DEtools** to make plots.

It is often informative to find the points in the (x, y) plane where the direction field is parallel to the x-axes, *i.e.*, when $y' = 0$, or is parallel to the y-axes *i.e.*, when $x' = 0$. Observe that $x' = 0$ when $-xy = 0$ and $y' = 0$ when $xy - y = 0$. These curves are called the *nullclines* for the system. The next Maple V segment illustrates how to draw these nullclines. Note that $x' = 0$ on each coordinate axes, and $y' = 0$ when $y = 0$ and $x = 1$. See Figure 65

```
> eq := -x*y=0,x*y-y=0;

                        eq := - x y = 0, x y - y = 0

> plt := plots[implicitplot]({eq},x=-1/2..3,y=-1..3):";
```

We now prepare to plot the direction field for the system, by using procedures from the **DEtools** package. Issue the **with** command. See Figure 66.

```
> with(DEtools:

> plt1 := DEplot2([deq],[x,y],t=0..1,x=-1/2..3,y=-1/2..3):";
```

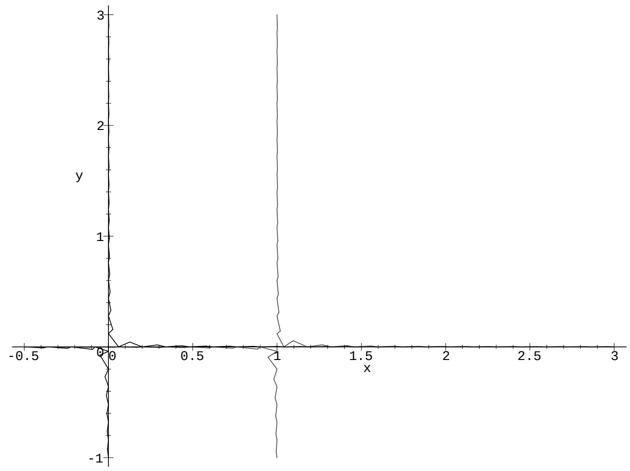

Figure 65: Nullclines for $x' = -xy$, $y' = xy - y$

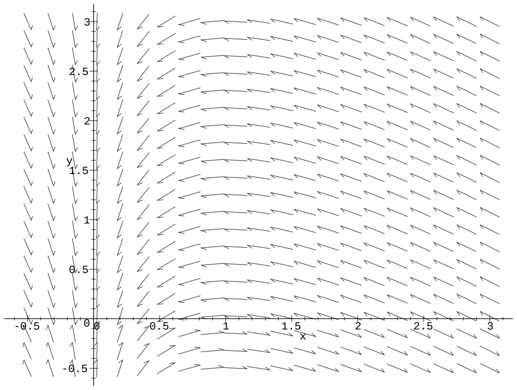

Figure 66: Direction Field of x'=-xy,y'=xy-y

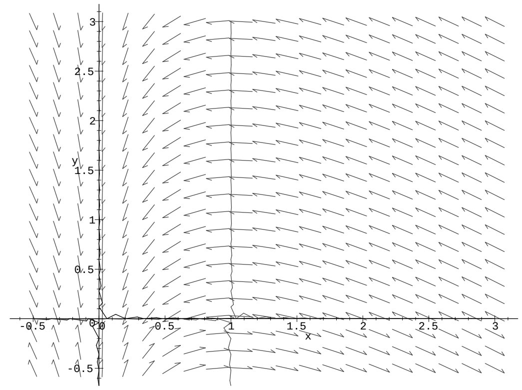

Figure 67: Nullclines for the system

A plot of the nullclines and direction field on the same graph is given in Figure 67. Observe that the direction field along the nullclines is parallel to one of the coordinate axis.

```
> plots[display]({plt,plt1});
```

The procedure **DEplot2** will now be used to obtain the solution curve in three dimensional space. Control of which variables are plotted is achieved by assigning the **scene** option. In Figure 68 a plot of the solution curve, satisfying the initial value problem $x(0) = 2.5$, $y(0) = 0.1$, in (t, x, y) space is obtained by using **scene = [t,x,y]**.

```
> DEplot2([deq],[x,y],0..4,{[x(0)=2.5,y(0)=0.1]},x=-1/2..3,

> y=-1/2..3,scene = [t,x,y],axes = normal);
```

In Figure 69 **DEplot2** with **scene =[x,y]** has been used to plot the solution curve that is shown in Figure 68 in the (x, y)-plane along with the direction field.

```
> DEplot2([deq],[x,y],0..10,{[x(0)=2.5,y(0)=0.1]},x=-1/2..3,y=-1/2..3,

> scene = [x,y]);
```

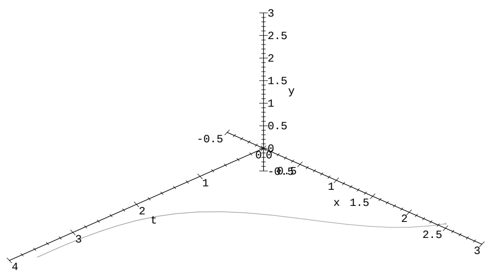

Figure 68: Solution for $x' = -xy, y' = xy - y$ in (t, x, y)-space

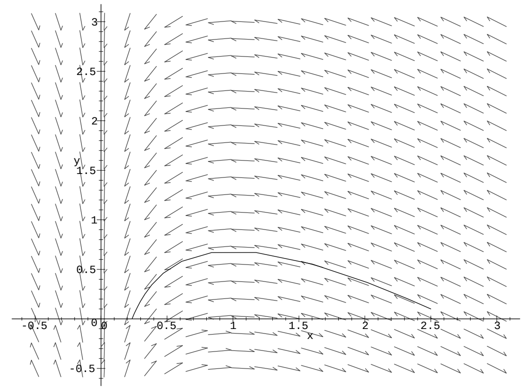

Figure 69: Direction field along with solution

When you have a solution to a system of equations with $x = \phi(t)$, and $y = \psi(t)$ you can also make plots of each of these curves with **DEplot2** by using **scene = [t,x]** and **scene = [t,y]** respectively. See Figures 70 and 71.

```
> DEplot2([deq],[x,y],0..10,{[x(0)=2.5,y(0)=0.1]},x=-1/2..3,y=-1/2..3,

> scene = [t,x]);

> DEplot2([deq],[x,y],0..10,{[x(0)=2.5,y(0)=0.1]},x=-1/2..3,y=-1/2..3,

> scene = [t,y]);
```

Systems Resulting From Mathematical Models

In the preceding section we studied population models that involved one species in isolation. Such models lead to a single first order differential equation. When we have the ability to use more than one differential equation at a time we can introduce models that involve more than one species.

Example 9.6.2 In this example we examine a system of differential equations that are derived from what is known as *predator-prey interaction*. Let $x(t)$, and $y(t)$ denote the population of a predator species and a prey species respectively. The predator-prey model assumes that $(x(t), y(t))$ satisfy the system:

$$x' = (-a + by)x, \;\; y' = (c - dx)y,$$

where a and c are positive numbers that are the decay (or death) and growth coefficients of each in the absence of the other species. It is assumed that the number of predator-prey encounters is proportional to the population of each. Thus b and d measure, respectively, predator efficiency in converting food (the prey) into fertility and the

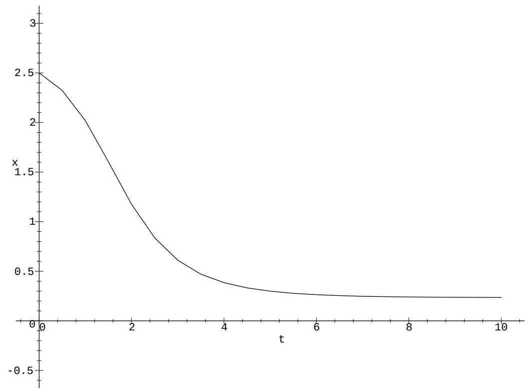

Figure 70: Solution for $x' = -xy$, $y' = xy - y$ in (t, x)-space

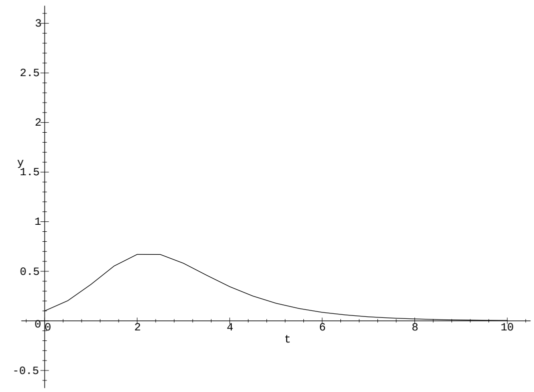

Figure 71: Solution for $x' = -xy$, $y' = xy - y$ in (t, y)-space

the probability that an encounter removes one of the prey. Analyze the solution space for values of the parameter given by $a = 1$, $b = \frac{1}{100}$, $c = 2$, and $d = \frac{2}{25}$.

Solution: First enter the equation into a Maple V session.

```
> deq := diff(x(t),t)=(-1+y(t)/100)*x(t),

> diff(y(t),t)=(2-2*x(t)/25)*y(t);
```

$$deq := \frac{d}{dt}x(t) = \left(-1 + \frac{y(t)}{100}\right)x(t),\ \frac{d}{dt}y(t) = \left(2 - \frac{2\,x(t)}{25}\right)y(t)$$

Note that if there is no prey, *i.e.*, that $y(t)$ equal zero, then the system reduces to a single differential equation

$$x' = -x$$

and the predators die off exponentially, since there is no food (the prey). Whereas, if there are no predators then the single equation is

$$y' = 2y,$$

and the prey explode exponentially and will soon exhaust their food supply.

It is usually productive to find the equilibrium points,

```
> equilibrii := solve({rhs(deq[1]),rhs(deq[2])},{x(t),y(t)});

         equilibrii := {y(t) = 0, x(t) = 0}, {y(t) = 100, x(t) = 25}
```

The last Maple V output tells us that there are two equilibria points. An equilibrium $(0, 0)$ means that if there are no predators and no prey at a given time then there never will be. The equilibrium at $(25, 100)$ means that if it ever happens that there are 25 predators and 100 prey, then there will *aways* be that number. It is more interesting to look at other solutions. Some plots using **DEplot2** of solution curves in phase space ((x, y)- space), (t,x) space, and (t,y) space for initial conditions

$$x(0) = 8, y(0) = 100;\ \ x(0) = 12, y(0) = 100;\ \text{ and } x(0) = 18, y(0) = 100,$$

are shown in Figures 72,73, and 74.

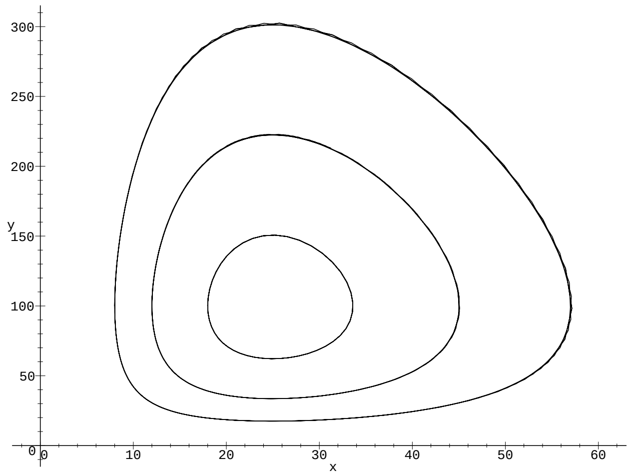

Figure 72: Phase space for $x' = (-1 + y/100)x$, $y' = (2 - 2x/25)$

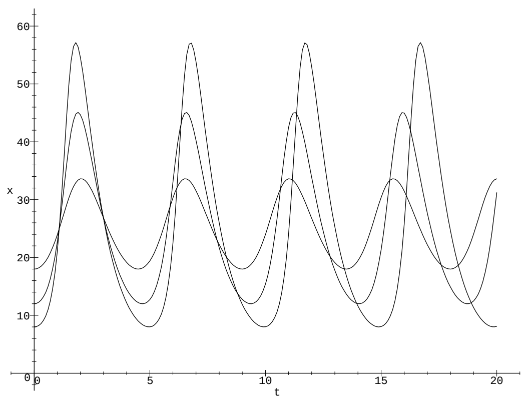

Figure 73: Solutions for $x' = (-1 + y/100)x$, $y' = (2 - 2x/25)$ in (t, x)-space

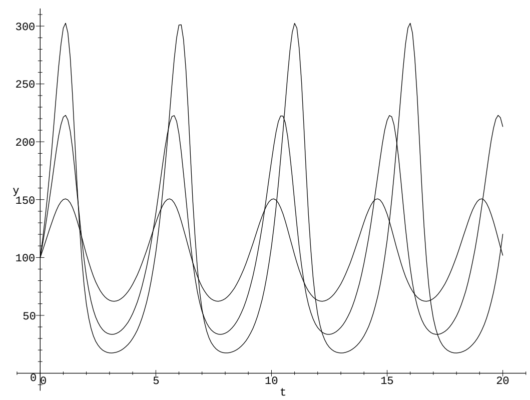

Figure 74: Solutions for $x' = (-1 + y/100)x$, $y' = (2 - 2x/25)$ in (t, y)-space

```
> with(DEtools):
> DEplot2([deq],[x,y],t=0..20,{[x(0)=8,y(0)=100],[x(0)=12,

> y(0)=100],[x(0)=18,y(0)=100]},arrows = NONE,x=0..60,y=0..300,

> scene = [x,y],stepsize = 0.1);

> DEplot2([deq],[x,y],t=0..20,{[x(0)=8,y(0)=100],[x(0)=12,

> y(0)=100],[x(0)=18,y(0)=100]},arrows = NONE,x=0..60,y=0..300,

> scene = [t,x],stepsize = 0.1);
```

```
> DEplot2([deq],[x,y],t=0..20,{[x(0)=8,y(0)=100],[x(0)=12,

> y(0)=100],[x(0)=18,y(0)=100]},arrows = NONE,x=0..60,y=0..300,

> scene = [t,y],stepsize = 0.1);
```

Populations that rise and fall as in this example exhibit a form of balance in the sense that both survive.

Example 9.6.3 A simple pendulum consists of a bob of mass m hanging on a (assumed to be massless) rigid rod of fixed length L firmly attached to a horizontal support. The pendulum is in equilibrium when the bob and rod are aligned with the local vertical and at rest. let $x(t)$ denote the angle that the rod makes with the vertical, let $y(t) = x'(t)$ be angular velocity. Then it can be shown that $(x(t), y(t))$ satisfies the equations

$$x' = y,\ y' = -\frac{g}{L}\sin x - \frac{c}{m}y,$$

where g is the acceleration due to gravity, and c is a constant due to friction. Suppose that $\frac{g}{L} = 1$, and $\frac{c}{m} = 0.2$, plot the phase plane with the direction field, along with solution curves that have initial values

$$x(0) = 0, y(0) = 2;\ x(0) = 0, y(0) = 2.5;\ x(0) = 0, y(0) = 3;\ \text{and } x(0) = 0, y(0) = 3.5.$$

Solution: Enter the differential equation and the initial values into a Maple V session. Then apply **DEplot2**.

```
> deq := diff(x(t),t)=y(t),diff(y(t),t)=-sin(x(t))-0.2*y(t);
```

$$deq := \frac{d}{dt}x(t) = y(t), \frac{d}{dt}y(t) = -\sin(x(t)) - 0.2y(t)$$

The Maple V command that creates the direction field along with the solution curves satisfying the given initial conditions in the (x, y)- plane is shown in Figure 75 and is written below.

```
> inits := {[x(0)=0,y(0)=2],[x(0)=0,y(0)=2.5],[x(0)=0,y(0)=3],

> [x(0)=0,y(0)=3.5]};
          inits := {[x(0) = 0, y(0) = 2], [x(0) = 0, y(0) = 2.5],

                [x(0) = 0, y(0) = 3], [x(0) = 0, y(0) = 3.5]}
> DEplot2([deq],[x,y],-10..20,inits,x=-10..10,y=-6..6,scene =[x,y],
> stepsize = 0.1);
```

If you wish to see the behavior of the same solution curves plotted with x vs t, then the next maple V command creates Figure 76.

```
> DEplot2([deq],[x,y],-10..20,inits,x=-10..10,scene =[t,x],

> stepsize = 0.1);
```

As was mentioned in the statement of this example the pendulum has two equilbria: when the bob and rod are aligned with the vertical and at rest. With the coordinates used here this means the points with coordinates $(0, 0)$ and $(\pi, 0)$. These represent points such that $x = 0$, which means the rod and bob are hanging straight down and are not moving, and the point with $x = \pi$, which means the rod and bob are balanced pointing straight up and are not moving. Observe that the right hand sides of the differential equation vanish simultaneously at infinitely many points: the points of the form $(n\pi, 0)$, for all integers n. You might think that this suggests that there are infinitely many equilibria. Indeed the differential equation does have infinitely many equilibria, but only two in the context of the equations as a model for a pendulum. Observe that the pendulum has the same position at the point $(0, y)$ as with any of the points $(2n\pi, y)$. In general the pendulum is in the same state when its coordinates are (x, y) or

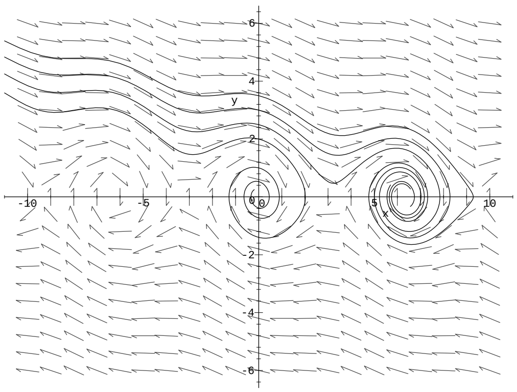

Figure 75: Solutions for pendulum in (x, y)-space

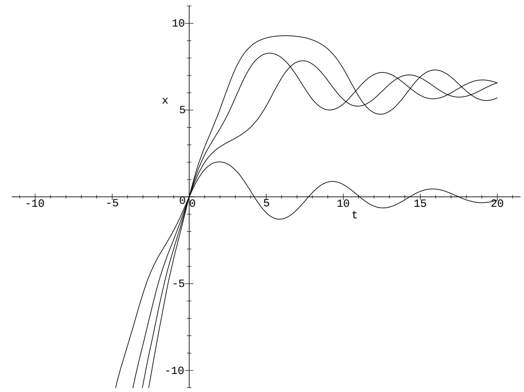

Figure 76: Solutions for pendulum in (t, x)-space

$(x + 2n\pi, y)$. Thus the only two distinct equilibria occur when the state of the pendulum is given by any pair of the form $(2n\pi, 0)$ ar $(n\pi, 0)$. The stability of these two points should be quite different, since a damped pendulum which is swinging back and forth might be expected to eventually settle down in the position in which it is hanging straight down. Moreover, if you try to balance a rod and bob straight up, the slightest push should send the rod and bob into motion which will ultimately come to rest hanging straight down. Figure 76 shows the x-component of three solutions for which initially $x = 0$ and the pendulum is set in motion at three different velocities: 2, 2.5, 3, and 3.5. The first solution spirals to the point $(0, 0)$ and from the graph it appears that $|x(t)|$ remains less than π, *i.e.*, the pendulum does not make a full revolution. But the other three orbits tend to the point $(2\pi, 0)$. This means that make one complete revolution before settling down to the stable equilbrium state of hanging straight down. The identification of points whose x-coordinates differ by integral multiples of 2π suggests that the phase space for the pendulum is actually a cylinder rather than the plane.

Exercises 9.6

1. In the following locate all equilbria and the use Maple V to make a direction field plot that includes all of the equlibria. In the vicinity of each equilibrium point fill in enough solution curves to determine whether solutions approach the equibrium or not.
   (a) $x' = 3x - 2y,\ y' = 2x - 2y$
   (b) $x' = 4x - 2y,\ y' = 8x - 4y$
   (c) $x' = -2x - y + 1,\ y' = y - 1$
   (d) $x' = y^2 - x^2,\ y' = y - 2x$
   (e) $x' = y^2 + x^2 - 4,\ y' = y^2 - x^2$

2. Red-Tail Hawks prey on the squirrel population on a certain college campus. Suppose that the number of squirrels, x, and the number of Red-Tailed Hawks, y, are governed by the equations

$$x' = 3x - xy,\ y' = -125y - 3xy.$$

   (a) Find all equilbrium points.
   (b) Plot the direction field for the (x, y)-plane for part of the first quadrant. Include several solution curves.
   (c) Plot the solution curves found in (b) in the (t, x) and (t, y) plane.
   (d) Discuss how the two populations can be expected to vary with time.

3. Consider the system
$$x' = -x + ay, \;\; y' = -x - y.$$
Make graphs that show how the character of the direction fields, and solution curves change as a varies from -1 to 1. For which values of a is there a sudden change in the nature of the solutions?

4. Consider an undamped pendulum
$$x' = y, y = -\sin(x).$$
   (a) Plot (in the (x, y)-plane) the solution curves which which are initially at the points:
   $$(-12, 1), \;\; (-12, 1.5), \;\; (-12, 2), \;\; \text{and } (-12, 3).$$
   Explain the different kind of motions of the pendulum correspond to closed and nonclosed curves.
   (b) Repeat part (a) with solutions that originate at
   $$(-6, b), \;\; (0, b), \;\; \text{and } (6, b),$$
   for some value of b.
   (c) Consider the closed curves which correspond to periodic motions of the pendulum originating at
   $$(1, 0), \;\; (1.5, 0), \;\; (2, 0), \;\; \text{and } (3, 0).$$
   Plot x versus t and estimate the period T of each solution. How does T depend on the initial position?
   (d) It can be shown that the solution which has initial value $x(0) = \alpha$, $y(0) = 0$ has a period equal to
   $$T = 4\int_0^{\pi/2} \frac{d\tau}{\sqrt{1 - k^2 \sin^2 \tau}}, \quad \text{where} \quad k = \sin(\frac{\alpha}{2}).$$
   Evaluate this integral numerically when $\alpha = 1, 1.5, 2,$ and 3. Compare your answer with the results of part (c).

9.7 Second-Order Linear Differential Equations

Let $a \neq 0$, and b be real numbers. A differential equation of the form
$$ax' + bx = 0$$
is called a first order linear differential equation with constant coefficients. It is an equation that can be solved either by the technique of separation of variables or by inspection. The general solution is
$$x(t) = Ce^{-\frac{b}{a}t}.$$
Observe that the coefficient of t in the exponent is a root of the first degree polynomial equation in λ,
$$a\lambda + b = 0.$$
The latter polynomial equation is called the characteristic equation for the differential equation
$$ax' + bx = 0.$$

Now let $a \neq 0$, b and c be real numbers. A differential equation of the form
$$ax'' + bx' + cx = 0$$

is called second order linear differential equation with constant coefficients. It will now be shown how to find the general solution to this equation. The solution should be a function which when added to a linear combination of its first and second derivative gives zero. Thus a solution of the form

$$\phi(t) = e^{\lambda t}$$

where λ is to be determined is sought. Substituting this function for x into the differential equation gives

$$a(\lambda^2 e^{\lambda t}) + b(\lambda e^{\lambda t}) + ce^{\lambda t} = 0.$$

Factoring $e^{\lambda t}$ from the left-hand side leads to the equation

$$(a\lambda^2 + b\lambda + c)e^{\lambda t} = 0,$$

which must be satisfied for all t. Recalling that $e^{\lambda t}$ can never be zero and thus can be divided out of the above equation leads to the second degree polynomial equation in λ,

$$a\lambda^2 + b\lambda + c = 0.$$

This equation is called the characteristic equation for the differential equation. If λ is a root of the characteristic equation then $x = e^{\lambda t}$ is a solution of the differential equation. The characteristic equation is easy to obtain: all one does is replace x by 1, x' by λ, and x'' by λ^2 in the differential equation.

Maple V can be used to make these calculations. Enter the differential equation:

```
> deq := a*diff(x(t),t$2)+b*diff(x(t),t)+c*x(t)=0;
```

$$deq := a\frac{d^2}{dt^2}x(t) + b\frac{d}{dt}x(t) + cx(t) = 0$$

Substitute $x(t) = e^{\lambda t}$ into this equation.

```
> e1 := subs(x(t)=exp(lambda*t),deq);
```

$$e1 := a\frac{d^2}{dt^2}e^{\lambda t} + b\frac{d}{dt}e^{\lambda t} + ce^{\lambda t} = 0$$

```
> e2  := simplify(e1);
```

$$e2 := a\lambda^2 e^{\lambda t} + b\lambda\, e^{\lambda t} + ce^{\lambda t} = 0$$

Now divide both sides by $e^{\lambda t}$ to obtain the characteristic equation. Note the usage of **expand** and **simplify**.

```
> ceq := simplify(expand(e2/exp(lambda*t)));
```

$$ceq := a\lambda^2 + b\lambda + c = 0$$

Each root to the characteristic equation leads to a solution of the differential equation. The following fact shows a way to obtain more solutions.

Let $\phi_1(t)$ and $\phi_2(t)$ be two solutions of the second order linear equation

$$ax'' + bx' + cx = 0$$

then if C_1, and C_2 are numbers

$$\phi(t) = C_1\phi_1(t) + C_2\phi_2(t)$$

is also a solution

This fact can be verified by direct substitution: Substitute $C_1\phi_1(t) + C_2\phi_2(t)$ for x in the differential equation and rearrange the terms to obtain

$$a(C_1\phi_1 + C_2\phi_2)'' + b(C_1\phi_1 + C_2\phi_2)' + c(C_1\phi_1 + C_2\phi_2) =$$

$$C_1(a\phi_1'' + b\phi_1' + c\phi_1) + C_2(a\phi_2'' + b\phi_2' + c\phi_2) = 0 + 0 = 0.$$

The next fact gives a condition which guarantees when two solutions of the equation can be used to generate solutions of all initial value problems.

> Let $\phi_1(t)$ and $\phi_2(t)$ be two solutions of the second order linear equation
>
> $$ax'' + bx' + cx = 0.$$
>
> Suppose that for some t_0 the inequality is true
>
> $$\Delta = \phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0) \neq 0.$$
>
> Let x_0, and x_0' be numbers. Then there are numbers C_1 and C_2 such that the function
>
> $$\phi(t) = C_1\phi_1(t) + C_2\phi_2(t)$$
>
> is a not only a solution of the differential equation but also satisfies the initial conditions $\phi(t_0) = x_0,\ \phi'(t_0) = x_0'$.

Since $\phi(t) = C_1\phi_1(t) + C_2\phi_2(t)$ is a solution for every pair of numbers C_1 and C_2, it follows that the statement in the box will be true if C_1 and C_2 can be found so that

$$\phi(t_0) = C_1\phi_1(t_0) + C_2\phi_2(t_0) = x_0,\ \phi'(t_0) = C_1\phi_1'(t_0) + C_2\phi_2'(t_0) = x_0'.$$

Recognizing that these two algebraic equations are linear in C_1 and C_2 enables one to write the solution

$$C_1 = \frac{x_0\phi_2'(t_0) - x_0'\phi_2(t_0)}{\Delta},\ C_2 = \frac{x_0'\phi_1(t_0) - x_0\phi_1'(t_0)}{\Delta}.$$

Example 9.7.1 Find the general solution to the second order differential equation

$$x'' - 3x' + 2x = 0.$$

Plot a few solution curves.

Solution: The characteristic equation is

$$\lambda^2 - 3\lambda + 2 = 0.$$

By factoring the right hand side it follow that the last equation is

$$(\lambda - 1)(\lambda - 2) = 0.$$

This means that two solutions are $\phi_1(t) = e^t$, and $\phi_2(t) = e^{3t}$. This gives the general solution since

$$\Delta = \phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0) = e^{t_0}(2e^{2t_0}) - e^{t_0}e^{2t_0} = e^{3t_0} \neq 0,$$

and we can solve every possible initial condition. We now will plot the solutions which satisfy the three sets of initial conditions

$$x(0) = -1, x'(0) = -1;\ x(0.5) = 0, x'(0.5) = 1;\ x(-0.5) = 0, x'(-0.5) = 1.$$

The general solution can be defined to Maple V as follows.

```
> gensol := C1*exp(t)+C2*exp(2*t);
```

$$gensol := C1\, e^t + C2\, e^{2\,t}$$

The constants $C1$, $C2$ are solved for the initial condition $x(0) = -1$, $x'(0) = -1$ by the following statement.

```
> Con1 := solve({subs(t=0,gensol)=-1,subs(t=0,
>          diff(gensol,t))=-1},{C1,C2});
```

$$Con1 := \{C2 = 0,\ C1 = -1\}$$

Substituting this result into the general solution gives the particular solution that satisfies the initial value problem.

```
> x1 := simplify(subs(Con1,gensol));
```

$$x1 := -e^t$$

The same Maple V steps can be followed to obtain the other two solutions.

```
> Con2 := solve({subs(t=0.5,gensol)=0,subs(t=0.5,diff(gensol,t))=1},{C1,C2});

                Con2 := {C2 = .3678794412, C1 = -.6065306596}
> x2 := simplify(subs(Con2,gensol));
```

$$x2 := \{C2 = 0,\ C1 = -1\}$$

```
> Con3 := solve({subs(t=-0.5,gensol)=0,subs(t=-
>  0.5,diff(gensol,t))=1},{C1,C2});

                Con3 := {C2 = 2.718281828, C1 = -1.648721271}

> x3 := simplify(subs(Con3,gensol));
```

$$x3 := -1.648721271e^t + 2.718281828e^{2.0t}$$

The plot of all three solutions is given in Figure 77

```
> plot({x1,x2,x3},t=-1..1,x=-4..4);
```

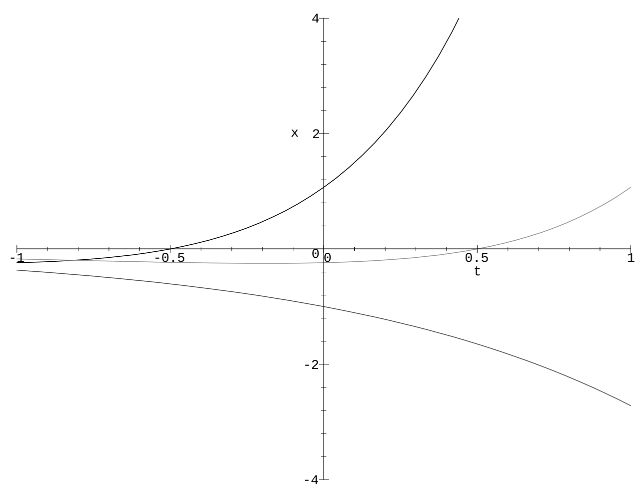

Figure 77: Solution curves for $x'' - 2x' + 2x = 0$

You should learn to solve simple second order linear differential with constant coefficients by hand. Probably there will be problems in which the computation becomes burdensome and thus you should also learn how to solve these equations exactly using Maple V. There follows a Maple V segment that indicates the relevent commands.

```
> deq := diff(x(t),t$2)-3*diff(x(t),t)+2*x(t)=0;
```

$$deq := \frac{d^2}{dt^2}x(t) - 3\,\frac{d}{dt}x(t) + 2\,x(t) = 0$$

The next step uses **dsolve** to find the general solution.

```
> gensol2 := dsolve(deq,x(t));
```

$$gensol2 := x(t) = \_C1\, e^t + \_C2\, e^{2\,t}$$

If you have a general solution, then you can use it to find the constants using **solve** from the initial conditions.

```
> constants := solve({subs(t=0,rhs(gensol2))= -1,subs(t=0,

> diff(rhs(gensol2),t))=-1},{_C1,_C2});
```

$$constants := \{\_C1 = -1, \_C2 = 0\}$$

Now the solution to the initial value problem is

```
> xx1 := subs(constants,rhs(gensol2));
```

$$xx1 := -e^t$$

This agrees with the answer which was obtained previously. You can also use **dsolve** to find the solution to the other initial value problem.

```
> xx2 := dsolve({deq,x(0.5)=0,D(x)(0.5)=1},x(t));
```

$$xx2 := x(t) = -0.6065306596 e^t + 0.3678794412 e^{2\,t}$$

This also agrees with the previously found solution. Note that the Maple V output is in the form of an equation, and thus you may need to use **rhs** when preparing to work with the solution. The solution to the third initial value problem can be found in the same way and is left as an exercise.

The Characteristic Equation

Since the characteristic equation

$$a\lambda^2 + b\lambda + c = 0$$

is so important to solving the differential equation

$$ax'' + bx' + cx = 0,$$

it will now be analyzed. The graph of the second degree polynomial

$$y = a\lambda^2 + b\lambda + c$$

is parabola which is concave up or concave down depending on the sign of a. There are thus three possibilities for the roots of the equation. See Figure 78 for the three possibilties that can occur when $a > 0$. The graphs for $a < 0$ are similar except are concave down.

Inspection of Figure 78 indicates that the roots of the characteristic will be one of the following:

1. two distinct real roots, r_1, and r_2 (when the vertex of the parabola is below the λ-axis),
2. one double root, r, (when the vertex of the parabola is tangent to the λ-axis,
3. a pair of complex conjugate roots (when the vertex of the parabola is above the λ-axis.

To see this algebraically we simply solve the quadratic equation

$$a\lambda^2 + b\lambda + c = 0.$$

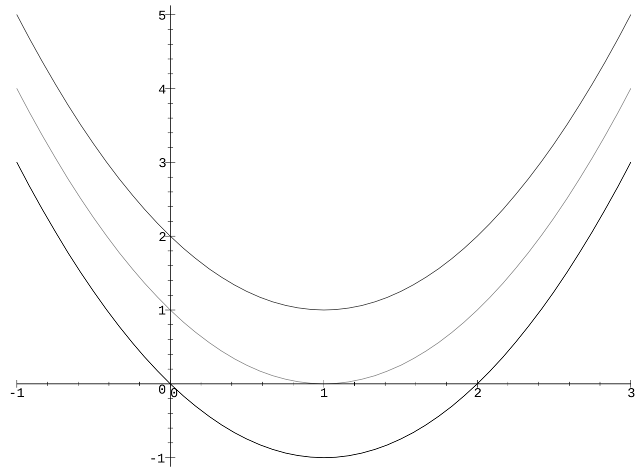

Figure 78: Possible positions of $y = a\lambda^2 + b\lambda + c$

The quadratic formula gives

$$\lambda = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Accordingly the roots will be real and distinct if the discriminant satisfies

$$b^2 - 4ac > 0.$$

Let

$$r_1 = \frac{b - \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{b + \sqrt{b^2 - 4ac}}{2a}.$$

The functions $\phi_1(t) = e^{r_1 t}$, and $\phi_2(t) = e^{r_2 t}$ are solutions. Observes that

$$\phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0) = (r_2 - r_1)e^{r_1 t_0 + r_2 t_0} \neq 0,$$

and hence

$$\phi(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

is a general solution.

In the second case when the roots merge into a single root the discriminant must vanish

$$b^2 - 4ac = 0$$

and hence

$$r = -\frac{b}{2a}.$$

In this case we only get one solution, $\phi_1(t) = e^{rt}$, by the procedure that is under discussion. It turns out that another solution is given by $\phi_2(t) = te^{rt}$. This follows by direct substitution after recognizing that in this case the characteristic equation has the form

$$a(\lambda^2 - 2r\lambda + r^2) = 0$$

which implies that the differential equation is

$$a(x'' - 2rx' + r^2 x) = 0.$$

Thus the general solution is

$$\phi(t) = (C_1 + C_2 t)e^{rt},$$

since

$$\phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0) = e^{rt_0} \neq 0.$$

Finally, when the discriminant satisfies

$$b^2 - 4ac < 0,$$

the characteristic equation has complex conjugate roots

$$r_1 = \frac{b - i\sqrt{4ac - b^2}}{2a},\ r_2 = \frac{b + i\sqrt{4ac - b^2}}{2a},$$

which, for purposes of simplification, will be written as

$$r_1 = \alpha - i\beta,\ r_2 = \alpha + i\beta.$$

Two solutions are then given by

$$\psi_1(t) = e^{(\alpha - i\beta)t)} = e^{\alpha t}e^{-i\beta t},\ \psi_t(t) = e^{(\alpha + i\beta)t)} = e^{\alpha t}e^{i\beta t}.$$

These solutions are not so pleasing, since they are complex valued, and, in this course, we are interested only in real-valued solutions. Recall that

$$\cos\beta t = \frac{e^{\beta t} + e^{-\beta t}}{2},\ \text{and} \sin\beta \mathrm{t} = \frac{\mathrm{e}^{\beta \mathrm{t}} - \mathrm{e}^{-\beta \mathrm{t}}}{2\mathrm{i}}.$$

This means that the functions defined by

$$\phi_1(t) = \frac{\psi_1(t) + \psi_2(t)}{2} = e^{\alpha t}\cos\beta t,\ \phi_2(t) = \frac{\psi_1(t) - \psi_2(t)}{-2i} = e^{\alpha t}\sin\beta t$$

are solutions and are real valued. Furthermore,

$$\phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0) = \beta e^{2\alpha t_0} \neq 0.$$

Thus a general solution for this case is

$$\phi(t) = e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t).$$

Oscillations

Suppose a mass m is attached to the end of a (massless) spring, the other end of which is attached to a solid horizontal beam. A coordinate system is established along the spring's axis and when the spring-mass configuration is in equilibrium the coordinate for the mass is zero and measures positive in the downward direction. It is assumed that the restoring force for the spring obeys Hooke's Law which means that if the mass is displaced to the point with coordinate x then the force is given by

$$F = kx,$$

for a contant k. In the absence of damping Newton's Second Law implies that the position $x(t)$ of the mass satisfies the differential equation

$$mx'' = -kx.$$

Sometimes it is assumed that the spring has damping which is proportional to the velocity of the mass and acts in a direction which is opposite to the motion. In this case $x(t)$ satisfies the differential equation

$$mx'' = -cx' - kx.$$

Example 9.7.2 Let the differential equation for a certain spring-mass system be

$$x'' + 4x = 0.$$

1. The mass is released from rest at a distance 2 units below the equilibrium position. Find a formula that gives the position of the mass as a function of time.
2. The mass is set in motion with a velocity of -3 ft/sec from a point a distance 2 units below the equilibrium position. Find a formula that gives the position of the mass as a function of time.
3. Express each of the solutions in the form
$$x(t) = A\cos(\omega t + \theta).$$
4. Make a Maple V plot of both solutions on the same graph over a time period equal to twice the period of the solutions.

Solution: The characteristic equation for the differential equation is
$$\lambda^2 + 4 = 0.$$
The characteristic roots are $\pm 2i$. In terms of the discussion above this means that the roots of the characterisic equation are complex with $\alpha = 0$, and $\beta = 2$. More precisely, one says that the roots are *pure imaginary.* The general solution is
$$\phi(t) = C_1 \cos 2t + C + 2\sin 2t.$$
If the mass is released from rest at a point 2 ft below the equilibrium position the solution must satisfy the initial conditions
$$\phi(0) = 2,\ \phi'(t) = 0.$$
The equations that determine C_1 and C_2 are
$$\phi(0) = C_1\cos(0) + C_2\sin(0) = C_1 = 2$$
$$\phi'(0) = -2C_1\sin(0) + C_2\cos(0) = C_2 = 0.$$
The solution is
$$\phi_1(t) = 2\cos(2t).$$
The solution in the second problem satisfies
$$\phi(0) = C_1\cos(0) + C_2\sin(0) = C_1 = 2$$
$$\phi(0) = -2C_1\sin(0) + C_2\cos(0) = C_2 = -3.$$
The second solution is
$$\phi_2(t) = 2\cos(2t) - 3\sin(2t).$$
The first solution is already in the form
$$x(t) = A\cos(\omega t + \theta),$$
with $A = 2$ and $\theta = 0$. Recall the addition formula for cos,
$$\cos(A + B) = \cos A\cos B - \sin A\sin B.$$
This means the second solution can be written as
$$\phi_2(t) = 2\cos(2t) - 3\sin(2t) = \sqrt{13}(\cos(2t)\frac{2}{\sqrt{13}} - \sin(2t)\frac{3}{\sqrt{13}}) = \sqrt{13}\cos(2t + \theta),$$
where $\tan(\theta) = \frac{\sin\theta}{\cos\theta} = 3/2$. Thus $\theta = \arctan(3/2)$ which is approximately equal to

```
> theta := evalf(arctan(3/2));
```

$$\theta := 0.9827937232$$

To check the accuracy the following can be used to check the result:

```
> evalf(sqrt(13)*cos(theta)); evalf(sqrt(13)*sin(theta));
```

```
2.000000000
```

```
2.999999999
```

Thus the answer checks to 8 decimal places. The example will be completed by giving a plot of the two solutions. The period of each solution is π, so the plot is over a time length of 2π. See Figure 79,

```
> plot({2*cos(2*t),2*cos(2*t)-3*sin(2*t)},t=0..2*Pi);
```

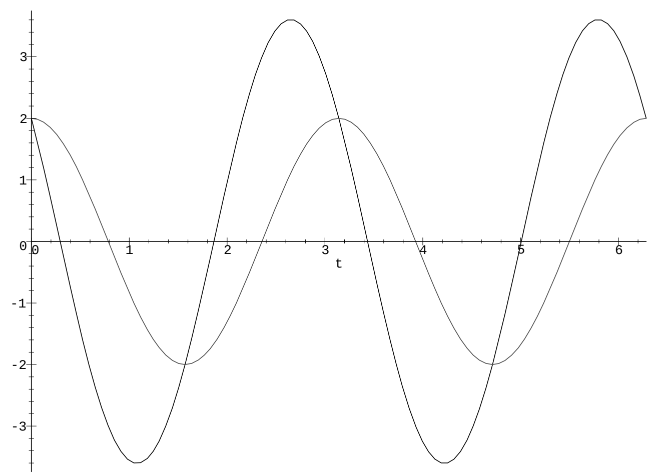

Figure 79: Two solutions of $x'' + 4x = 0$.

Example 9.7.3 Let the differential equation for a certain spring-mass system with damping is

$$x'' + 0.1x' + 0.2x = 0.$$

1. The mass is released from rest at a distance 2 units below the equilibrium position. Find a formula that gives the position of the mass as a function of time.

2. The mass is set in motion with a velocity of -3 ft/sec from a point a distance 2 units below the equilibrium position. Find a formula that gives the position of the mass as a function of time.

3. Express each of the solutions in the form

$$x(t) = Ae^{\alpha t}\cos(\omega t + \theta).$$

4. Make a Maple V plot of both solutions on the same graph over a time interval [0, 30].

Solution: The characteristic equation for the differential equation is

$$\lambda^2 + 0.1\,\lambda + 0.2.$$

Maple V will ease the burden of some the calculations.

```
> eq := lambda^2+0.1*lambda+0.2=0;
```

$$eq := \lambda^2 + 0.1\lambda + 0.2 = 0$$

The characteristic roots can be found using **solve**.

```
> sol := solve(eq,lambda);

sol :=  - .05000000000 + .4444097209 I,
                  - .05000000000 -.4444097209 I
```

These roots are complex conjugate with $\alpha = -0.05$ and $\beta = .4444097209$. This means that the general solution is

$$\phi(t) = e^{.05t}(C_1 \cos(.4444097209t) + C_2 \sin(.4444097209t).$$

Note that α is the *real part* and β is the *imaginary part* of the complex number $\alpha + i\beta$. This suggests the use of the Maple V commands **Re** and **Im** to find the general solution.

```
> phi := exp(Re(sol[1])*t)*(C1*cos(Im(sol[1])*t)+C2*sin(Im(sol[1])*t));
```

$$phi := e^{-0.05000000000t}\,(C1\,\cos(0.4444097209t) + C2\,\sin(0.4444097209t))$$

The constants are found for the first initial value problem.

```
> cons1 := solve({subs(t=0,phi)=2,subs(t=0,diff(phi,t))=0},{C1,C2});

                    cons1 := {C1 = 2., C2 = .2250175802}
```

The solution to the first initial value problem is obtained by substituting these values for the C's.

```
> xx1 := subs(cons1,phi);
```

$$xx1 := e^{-0.05000000000t}\,(2.0\cos(0.4444097209t) + 0.2250175802\sin(0.4444097209t))$$

The last Maple V output is the desired solution. The solution to the second initial value problem follows similarly.

```
> cons2 := solve({subs(t=0,phi)=2,subs(t=0,diff(phi,t))=-3},{C1,C2});

                    cons2 := {C1 = 2., C2 = -6.525509825}

> xx2 := subs(cons2,phi);
```

$$xx2 := e^{-0.05000000000t}\,(2.0\cos(0.4444097209t) - 6.525509825\sin(0.4444097209t))$$

The next thing to do is to express the solution in the form

$$Ae^{\alpha t}\cos(\omega t + \theta).$$

The next Maple V segment finds A and θ for the first solution.

```
> A1 := subs(cons1,sqrt(C1^2+C2^2));

                              A1 := 2.012618422


> tan1 := subs(cons1,-C2/C1);

                             tan1 := -.1125087901
> theta1 := arctan(tan1);
```

$$\theta 1 := -0.1120376427$$

Thus the first solution can be written as

$$\phi_1(t) = 2.012618422e^{-.05t}\cos(0.4444097209t - 0.1120376427)$$

The second solution can be treated similarly.

```
> A2 := subs(cons2,sqrt(C1^2+C2^2));
```

```
A2 := 6.825121133
```

```
> tan2 := subs(cons2,-C2/C1);
```

```
tan2 := 3.262754913
```

```
> theta2 := arctan(tan2);
```

$$\theta 2 := 1.273396582$$

This means that the second solution can be written as

$$\phi_2 = 6.825121133e^{-.05t}\cos(0.4444097209t + 1.273396582)$$

The next two maple V commands represent checks of the correctness of the preceding calculations.

```
> expand(A1*cos(Im(sol[1])*t+theta1));
       2.000000000 cos(.4444097209 t) + .2250175803 sin(.4444097209 t)
> expand(A2*cos(Im(sol[1])*t+theta2));
       1.999999997 cos(.4444097209 t) - 6.525509826 sin(.4444097209 t)
```

It follows that, except for roundoff error, the above calculations are correct. See Figure 80 for the plot of the two solutions.

```
> plot({xx1,xx2},t=0..30);
```

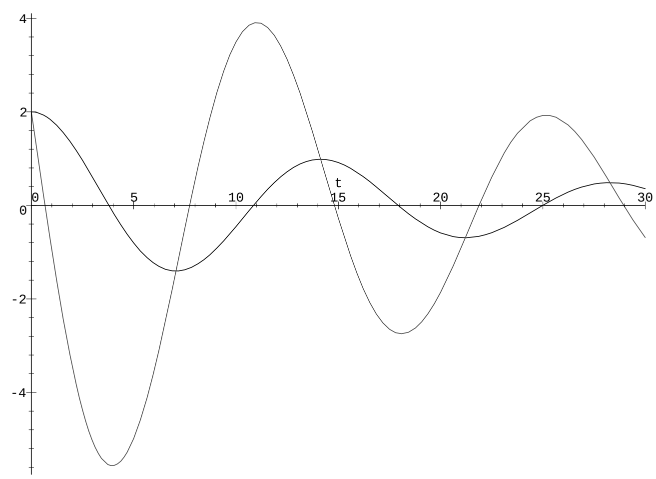

Figure 80: Two solutions of $x'' + 0.1x' + 0.2x = 0$.

Exercises 9.7

1. Find the characteristic equation and use it to obtain the general solution for each of the following second order linear differential equations.

   (a) $x'' - 3x' - 10x = 0$

   (b) $x'' + 10x' + 25 = 0$

   (c) $x'' + 4x' + 13x = 0$

2. In each of the following problems the motion of a mass attached a spring is described by an initial value problem.

   (a) Solve each of the problems.

   (b) Plot the solution in the specified interval.

   (c) Find the maximum of the solution in the specified interval.

   (d) When possible express each solution in the form

   $$Ae^{\alpha t}\cos(\omega t + \theta).$$

   (a) $x'' + 15 = 0,\ x(0) = -2, x'(0) = 5;\ 0 \le t \le 5$

   (b) $x'' + 14.9x = 0,\ x(0) = -2, x'(0) = 5;\ 0 \le t \le 5$

   (c) $x'' + 5.1x' + 6x = 0,\ x(0) = -1.2, x'(0) = 3;\ 0 \le t \le 4$

   (d) $x'' + 3.9x' + 18.73x = 0,\ x(0) = 3, x'(0) = 2;\ 0 \le t \le 4$

3. In order to examine changes in the amplitude of an oscillation make Maple V plots of the functions

   $$\phi(t) = A\cos(t+1), \quad \text{for} \quad A = 0, 1/2, 1, 2,$$

   on the same graph.

4. In order to examine how changes phase shift changes the graph make Maple V plots of the functions

   $$\phi(t) = \cos(t+\theta), \quad \text{for} \quad \theta = 0, 1/2, 1, 2,$$

   on the same graph.

5. In order to examine how changes phase shift changes the graph make Maple V plots of the functions

   $$\phi(t) = \cos(\omega t+1), \quad \text{for} \quad \omega = 0, 1/2, 1, 2,$$

   on the same graph.