8 Applications That Use The Definite Integral

In Chapter 3 it was shown that the concept of the definite integral could be used to calculate the following quantities: the distance traveled by a moving object when its instantaneous velocity is known, to evaluate the the area under a curve given by a nonnegative continuous functon defined on a closed interval, and, more generally, to compute the total change in a quantity defined as a function which varies over a closed interval. In the preceding chapter various methods that can be used to evaluate integrals were considered. In this chapter further examples are given to illustrate how integrals can be used to solve a wide variety of problems. In each application the fact that a definite integral is defined as the limit of Riemann Sums of the form

$$\sum_{k=1}^{n} f(c_k)\Delta t_k,$$

where the c_k are points from a partition $\{a = t_0 < t_1 < \cdots < t_n = b\}$ and $\Delta t_k = t_k - t_{k-1}$ is used.

8.1 The Area between Two Curves

The problem of finding the area between two curves, will be reviewed, in order to illustrate this approach.

Example 8.1.1 Find the area enclosed by the parabola $y = 4 - x^2$ and the straight line $y = 1 - 2x$.

Solution: The first step in a problem which involves finding the area bounded between two curves is to plot the graph of the two curves simultaneously.

Using Maple V we define the two functions.

```
> f := x -> 4 - x^2;  g := x ->1-2*x;
```

$$f := x \mapsto 4 - x^2$$

$$g := x \mapsto 1 - 2\,x$$

Now plot both curves simultaneously. See Figure 25.

```
> Plt1 := plot({4-x^2,1-2*x},x=-1.5..3.5): plots[display](Plt1);
```

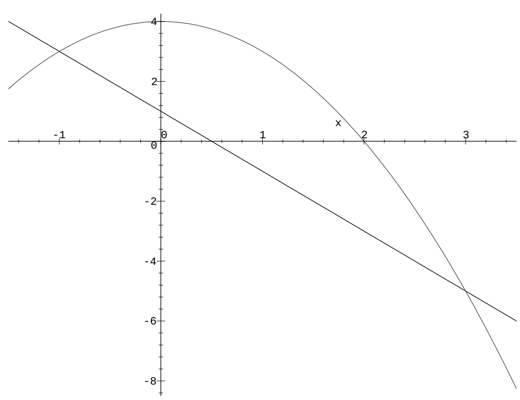

Figure 25: Curves $f(x) = 4 - x^2$ and $g(x) = 1 - 2x$

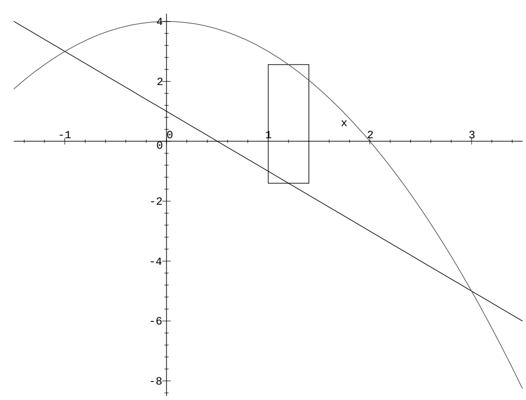

Figure 26: Curves with an element of area

Recall that to find the area of a region like the one shown in Figure 25 one takes a limit of Riemann Sums that are defined over a partition of the interval with left-hand endpoint the $x-$ coordinate of the left-hand point of intersection of the curves and with right-hand endpoint the $x-$ coordinate of the right-hand point intersection of the two curves. The summand in this sum is of the form $\Delta A_k = (f(c_k) - g(c_k))\Delta x$. For problems involving the finding of areas a typical term in the summand is called an *element of area.* It is usually a good idea to plot a picture of the region along with a typical summand from the Riemann Sum or, what is the same, of a typical element of area. See Figure 26.

The following Maple V segment can be used to plot Figure 26.

```
> Plt2 := plots[polygonplot]([[1,g(1.2)],[1.4,g(1.2)],
> [1.4,f(1.2)],[1,f(1.2)]]):
> plots[display]({Plt1,Plt2});
```

An approximation to the area bounded between the two curves is given by a Riemann Sum of with summand as above. Since by definition the Riemann Integral

$$\int_a^b (f(x) - g(x))\,dx$$

is a limit of Riemann Sums

$$\sum_{k=1}^{n} (f(c_k) - g(c_k))\Delta x_k,$$

it follows that the area between the two curves is equal to this definite integral with a and b to be determined as described above.

In order to find a and b, solve the equation

$$f(x) = 4 - x^2 = g(x) = 1 - 2x$$

for x. This is easily solved by hand by the following steps. Adding to both sides of the equation

$$4 - x^2 = 1 - 2x$$

leads to

$$x^2 - 2x - 3 = 0$$

Factoring the left-hand side of the last equation leads to

$$(x+1)(x-3) = 0$$

Consequently, the left-hand intersection point has coordinates $(-1, 3)$, and the right-hand intersection point has coordinates $(3, -5)$. The area, A, of the region is equal to the value of

$$A = \int_{-1}^{3} (f(x) - g(x))\,dx = \int_{-1}^{3} ((4 - x^2) - (1 - 2x))\,dx = \int_{-1}^{3} (3 + 2x - x^2)\,dx.$$

This integral is easily calculated by hand and we have:

$$A = 3x + x^2 - \frac{x^3}{3}\Big|_{-1}^{3} = (9 + 9 - 9) - (-3 + 1 + \frac{1}{3}) = \frac{32}{3}.$$

In more complicated problems you may need to use Maple V to calculate exactly or, if necessary, approximately, the intersection points and the value of the integral.

```
> sol := solve({y=f(x),y=g(x)},{x,y});
                    sol := {y = -5, x = 3}, {y = 3, x = -1}
```

Observe that the points of intersection have coordinates given by $(-1, 3)$, and $(3, -5)$, which agrees with the previous hand calculation. The value of the integral is given by

```
> Int(f(x)-g(x),x=-1..3): "= value(");
```

$$\int_{-1}^{3} 3 - x^2 + 2\,x dx = \frac{32}{3}$$

Therefore, the area between the two curves is $\frac{32}{3}$.

In the sections that follow the definite integral will be applied to solve what appears to be different types of problems. But each problem has much in common. In each problem one obtains a Riemann Sum over an interval that represents an approximation to the actual value of some quantity which which is to be computed. The summand of this Riemann Sum provides the integrand for a definite integral over the same interval which is equal to the exact value of this quantity.

Exercises 8.1

1. Find the area bounded between $f(x) = \cos x$ and $g(x) = \sin x$ and the interval $x = 0$, and $x = \frac{\pi}{4}$.
2. Find the area in the first quadrant bounded between the parabola $y = \sqrt{x}$, and the straight line $y = x - 3$.
3. Find the area in the first quadrant above the line $y = x$ and below the curve $y = \sin(x)$.
4. Find the area of the region bounded between the two curves $y = x^4 - 3x^3$ and $y = 2x^2$.

8.2 Applications To Geometry

In the previous section you saw how the definite integral is used to calculate the area of a region formed between to curves. In this section we illustrate how the definite can be used to compute the volume of certain solids and the length of an arc. The approach to finding a volume of a solid will be to think of the solid as approximated by small elements, each of which is so geometrically simple that its volume can be calculated directly. Next the volumes of each of these elements are added to obtain a Riemann Sum. The limit of such Riemann Sums give the volume.

Volumes of Given Cross-Section

When calculating the volume of a solid using Riemann Sums, slice the solid into thin pieces in which the geometry is so simple that the volume can be estimated.

Example 8.2.1 The Great Pyramid of Egypt has a square base with side 755 feet long and height 410 feet. Compute the volume of the Great Pyramid in cubic feet.

Solution:

Just as when finding areas of regions the first step in finding volumes is to plot the graph of the solid. The following Maple V segment creates the plot of the Great Pyramid that is shown in Figure 27.

```
> face1 := [[755,0,0],[755,755,0],[755/2,755/2,410],[755,0,0]]:

> face2 := [[755,755,0],[0,755,0],[755/2,755/2,410],[755,755,0]]:

> face3 := [[0,755,0],[0,0,0],[755/2,755/2,410],[0,755,0]]:

> face4 := [[0,0,0],[755,0,0],[755/2,755/2,410],[0,0,0]]:

> BASE := [[755,0,0],[755,755,0],[0,755,0],[0,0,0],[755,0,0]]:
> pyr1 := plots[polygonplot3d]({face1,face2,face3,face4,BASE},
  axes=framed, > style=wireframe,orientation=[30,60]);
```

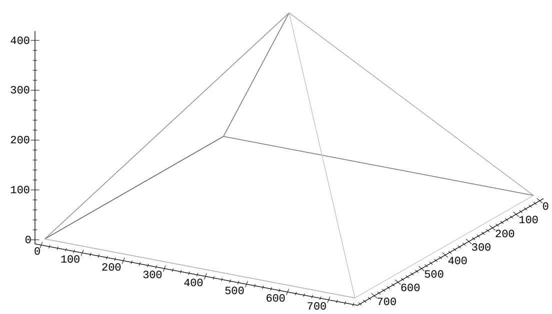

Figure 27: The Great Pyramid

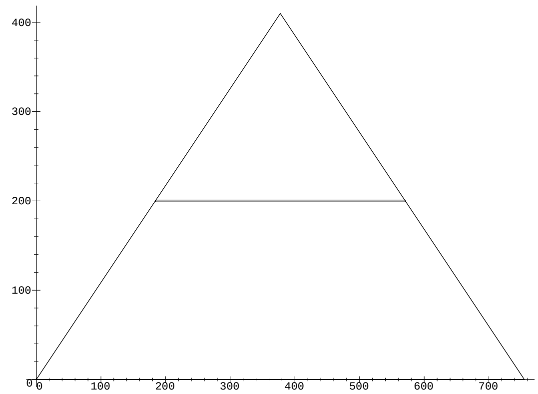

Figure 28: A Face ofthe Great Pyramid showing a cross section of an element of volume

Now think of the pyramid as being made up of layers parallel to the base. Each layer is a thin rectangular box with square base and with thickness Δz. Figure 28 illustrates a cross-section of a typical face and rectangular box. In order to see how to make a plot like in Figure 28, make a few calculations. Let s denote the length of the base of a typical layer, then the similar triangles shown in Figure 28 imply that

$$\frac{s}{755} = \frac{410 - z}{410},$$

where z denotes the height above the horizontal that the center of the layer lies. Solving for s one sees that the length of the rectangular box is given by the formula

$$s = 755 - \frac{755}{410}z.$$

The coordinates of the cross-sectional rectangle with center at height 200, and with $\Delta z = 2$ feet is determined as follows. Using a midpoint rectangle observe that, for example, the lower left-hand point has horizontal coordinate at $755/2 - s(200)/2$ and vertical coordinate 199. The other three vertices are found similarly and the following Maple V segment plots Figure 28. The first step is to define the function $s(z)$.

```
> s := z -> 755 - 755*z/410;
                          s := z -> 755 - 151/82 z

> plt1 := plots[polygonplot]([[0,0],[755,0],[755/2,410],[0,0]]):
> plt2 := plots[polygonplot]([[755/2-s(200)/2,199],[755/2+s(200)/2,199],
> [755/2+s(200)/2,201],[755/2-s(200)/2,201],[755/2-s(200)/2,199]]):
> plots[display]({plt1,plt2});
```

The element of volume which corresponds to the rectangle that occurs in Figure 28 is a rectangular box with vertices on the bottom square located at the four points $(755/2 - s(200)/2, 755/2 - s(200)/2, 199)$, $(755/2 + s(200)/2, 755/2 - s(200)/2, 199)$, $(755/2 + s(200)/2, 755/2 + s(200)/2, 199)$, and $(755/2 - s(200)/2, 755/2 + s(200)/2, 199$. The top rectangle has similar coordinates, except the z coordinate is 201. The following Maple V segment illustrates a plot of the pyramid with the central slice of this element of volume. See Figure 29.

```
> rect1 := plots[polygonplot3d]([[755/2-s(200)/2,755/2-s(200)/2,200],
>[755/2+s(200)/2,755/2-s(200)/2,200],[755/2+s(200)/2,755/2+s(200)/2,200],
>[755/2-s(200)/2,755/2+s(200)/2,200],[755/2-s(200)/2,755/2-s(200)/2,200]],
> style=patch):
> plots[display]({pyr1,rect1});
```

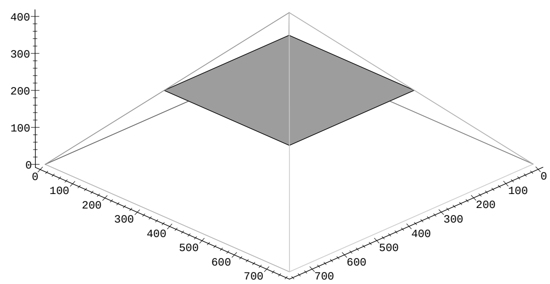

Figure 29: The Great Pyramid with one cross-sectional slice

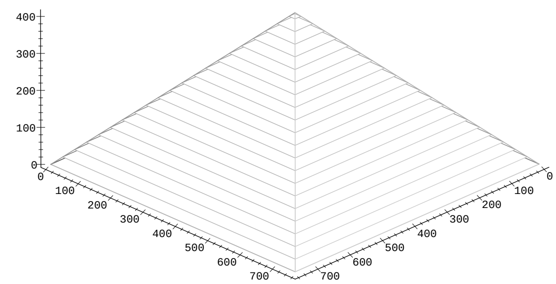

Figure 30: The Great Pyramid with numerous slices

In order to get an idea of how the pyramid can be approximated by thin rectangular boxes having square bases parallel to the base of the pyramid, construct a very simple Maple V procedure which gives the coordinates of the vertices of a square parallel to the base of the pyramid located a distance z above the base and having vertices on the edges of the pyramid.

```
> makerects := proc(z)
> [[755/2-s(z)/2,755/2-s(z)/2,z],[755/2+s(z)/2,755/2-s(z)/2,z],
> [755/2+s(z)/2,755/2+s(z)/2,z],[755/2-s(z)/2,755/2+s(z)/2,z],
> [755/2-s(z)/2,755/2-s(z)/2,z]];
> end;

makerects :=

proc(z)
    [[755/2-1/2*s(z),755/2-1/2*s(z),z],[755/2+1/2*s(z),755/2-1/2*s(z),z],
        [755/2+1/2*s(z),755/2+1/2*s(z),z],[755/2-1/2*s(z),755/2+1/2*s(z),z],
        [755/2-1/2*s(z),755/2-1/2*s(z),z]]
end
```

Using this procedure a plot of The Great Pyramid along with a number of these slices is given in Figure 30.

```
> for i from 0 to 20  do
> plt[i]:= plots[polygonplot3d](makerects(i*20,style=patch)):
> od:
> plots[display]({pyr1} union {seq(plt[i],i=0..20)});
```

The volume of a typical rectangle, or element, in this collection is equal to:

$$s^2 \cdot \Delta z = (755 - \frac{755}{410} c_i)^2 \Delta z_i.$$

Thus an approximation to the volume V of the pyramid has the form:

$$V \approx \sum_{i=1}^{n} (755 - \frac{755}{410} c_i)^2 \Delta z_i.$$

Taking the limit as n tends to ∞ and using the definition of the definite integral permits on to conclude that the volume is a number equal to the value of the integral

$$V = \int_0^{410} (755 - \frac{755}{410} z)^2 \, dz.$$

The integrand of this integral is a second degree polynomial and is easily calculated by hand, but when you evaluate the integral at the end points you will probably need a calculator. Maple V can make this calculation directly.

```
> Int((755-755/410*z)^2,z=0..410): "= value(");
```

$$\int_0^{410} \left(755 - \frac{151\,z}{82}\right)^2 dz = \frac{233710250}{3}$$

Thus the total volume of The Great Pyramid is $\frac{233710250}{3}$ cubic feet which is approximately

```
> evalf(rhs("));
```

$$.7790341667\ 10^8$$

cubic feet.

Example 8.2.2 Find the volume of a sphere.

Solution: Here you are asked to solve a problem in which you already know the answer; *i.e.* the volume of a sphere of radius a is $\frac{4}{3}\pi a^3$. This result can be derived using the Fundamental Theorem of Calculus. First write the equation of the sphere of radius a centered at the origin for use in a Maple V segment.

```
> eq1 := x^2 + y^2 +z^2 = a^2;
```

$$eq1 := x^2 + y^2 + z^2 = a^2$$

Use the Maple V procedure **implicitplot3d** to plot the sphere. See Figure 31 for the plot of the sphere when $a = 1$.

```
> plots[implicitplot3d](subs(a=1,eq1),x=-1..1,y=-1..1,z=-1..1,
> axes=boxed);
```

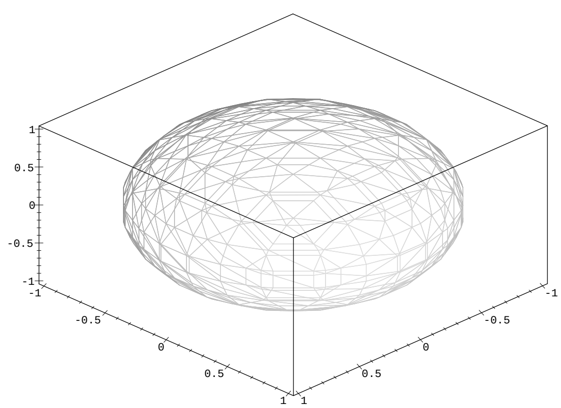

Figure 31: The Unit Sphere

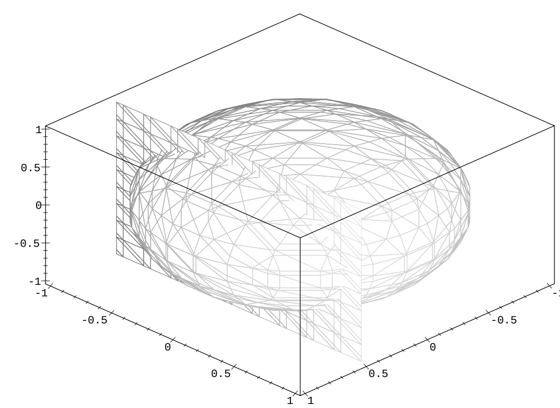

Figure 32: The Unit Sphere with a slice $x = 1/2$

A plot of the unit sphere along with the slice cut by the plane $x = 1/2$ is shown in Figure 32.

```
> plots[implicitplot3d]({subs(a=1,eq1),x=1/2},x=-1..1,y=-1..1,z=-

> 1..1,axes=boxed);
```

The intersection of a plane of the form $x = x_0$ and the sphere

$$x^2 + y^2 + z^2 = a^2$$

is a disk. To obtain information about this disk let us view the cross section of the sphere through $z = 0$. This the disk enclosed by the circle $x^2 + y^2 = a^2$. See Figure 33 for the case $a = 1$.

```
> plots[implicitplot](x^2+y^2=1,x=-1..1,y=-1..1,scaling=constrained);
```

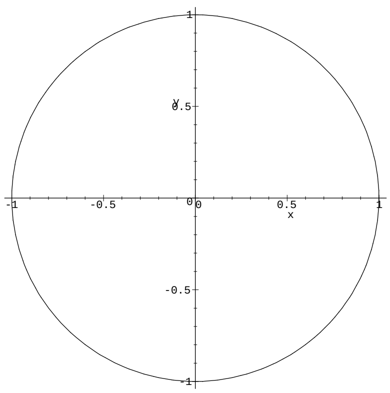

Figure 33: The projection of the Unit Sphere on the plane $z = 0$

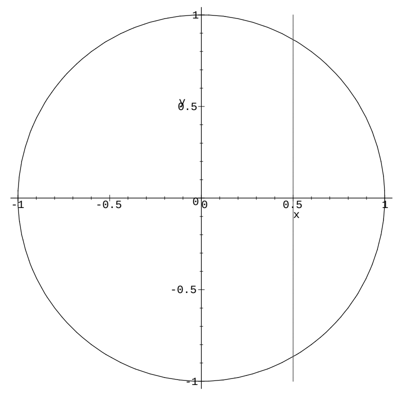

Figure 34: The projection with a slice $x = 1/2$

Now consider the slice through this disk that is cut out by the intersection of the plane $x = x_0$, with this disk. See Figure 34 for the case that $a = 1$ and $x_0 = 1/2$.

```
> plots[implicitplot]({x^2+y^2=1,x=1/2},x=-1..1,y=-1..1,
> scaling=constrained);
```

The radius of the disk which is cut from the sphere by the plane $x = 1/2$ is equal to one half of the vertical line segment formed by the intersection of the unit disk and the vertical line $x = 1/2$ and shown in Figure 34. The radius of the disk is $y = \sqrt{1 - (1/2)^2}$ in this case. In general, for arbitrary a and x_0 the radius is $\sqrt{a^2 - x_0^2}$. Using plain x instead of x_0, the following Maple V segment helps with these calculations:

```
> student[isolate](eq1,y^2);
```

$$y^2 = a^2 - x^2 - z^2$$

The radius for the cross section at a specific value of x is

```
> radius := sqrt(subs(z=0,rhs(")));
```

Now think of the sphere as approximated by thin cylinders with radius given as above an thickness Δx. The volume of one of these elements is

$$\Delta V_i = \pi \left(\sqrt{a^2 - x_i^2} \right)^2 \Delta x_i .$$

A Riemann Sum will consist of a sum of such volumes and in the limit as Δx_i tends to 0, the exact volume is equal to

$$V = \pi \int_{-a}^{a} (a^2 - x^2)\, dx.$$

The familiar formula for the volume of a sphere now follows.

```
> Int(Pi*radius^2,x=-a..a):"=value(");
```

$$\int_{-a}^{a} \pi \left(a^2 - x^2\right) dx = \frac{4\,\pi\,a^3}{3}$$

Volumes of Solids of Revolution

In the preceding section a solid was imagined to be sliced up into thin elements each of whose volume could be estimated. This is particularly easy to accomplish if the solid can be created by revolving a plane region about some line in space.

Example 8.2.3 The region bounded by the curve $y = xe^{-x}$ and the x-axis between $x = 0$ and $x = 2$ is revolved about the x-axis. Find the volume of the solid which is formed.

Solution: As usual it is a good idea to make some plots in order to visualize the problem. Define the function $f(x) = xe^{-x}$ in a Maple V session.

```
> f := x -> x*exp(-x);
```

$$f := x \mapsto xe^{-x}$$

The next Maple V segment plots the graphs of the function, the line segment from $(2, 0)$ to $(2, f(2))$, and a typical element of area for the region under consideration. See Figure 35.

```
> plt1 := plot(f(x),x=0..2):
> plt2  := plot([[2,0],[2,f(2)]]):
> element1 :=
> plots[polygonplot]([[1,0],[1.2,0],[1.2,f(1.1)],[1,f(1)]]):
> plots[display]({plt1,plt2,element1});
```

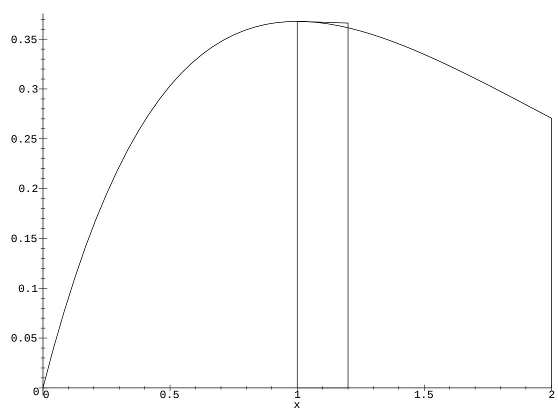

Figure 35: Region bounded by xe^{-x}, between $x = 0$ and $x = 2$

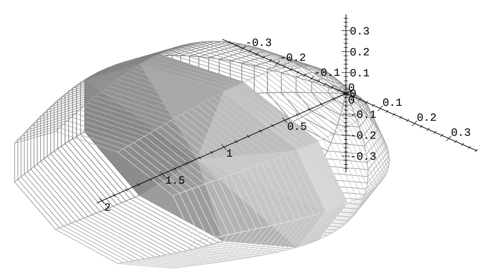

Figure 36: Solid formed by revolving region about x-axis

The solid is formed by revolving the region shown in Figure 35 about the x-axis. An element of volume is obtained by revolving an element of area for the region about the x-axis. The solid and a typical element of volume is shown in Figure 36. The Maple V procedure called **tubeplot** is used. The procedure is part of the **plots** package. The proper syntax for it is **tubeplot**(C,$<$ options $>$), where C is a set of space curves. In our example we assign the option **radius** to be the function

$$f(x) \cdot (H(1) - H(1.2)),$$

where H is the Heaviside Function, for the element of volume created by revolving the rectangle with base the line segment from $(1,0)$ to $(1.2,0)$ and height $f(1.1)$. This element of volume is a thin solid cylinder with radius $f(1.1)$ and thickness $\Delta x = 0.2$. The **radius** is assigned to be $f(x)$ for the **tubeplot** of the entire solid.

```
> radius := 'radius':

> plt3 :=plots[tubeplot]([x,0,0],x=0..2,radius =f(x)*(Heaviside(x-1)

> -Heaviside(x-1.2)),axes=normal,style=patchnogrid):

> plt4 := plots[tubeplot]([x,0,0],x=0..2,radius = f(x),
> style=wireframe):

> plots[display]({plt3,plt4});
```

One can think of approximating the volume of the solid by adding up the small cylinders with radius equal to $f(x)$ and thickness Δx. Such a cylinder has volume $\pi f(x)^2 \Delta x$. Terms like this appear as a summand in the sum obtained by partitioning the interval[0, 2] and writing a Riemann Sum. Therefore, the volume is

$$V = \pi \int_0^2 f(x)^2\, dx = \pi \int_0^2 x^2 e^{-2x}\, dx.$$

Calculate the volume exactly, as follows:

```
> Pi*Int(f(x)^2,x=0..2): "=value(");
```

$$\pi \int_0^2 x^2 \left(e^{-x}\right)^2 dx = \pi \left(-\frac{13\left(e^{-2}\right)^2}{4} + 1/4\right)$$

The volume to ten digits of accuracy is equal to

```
> evalf(rhs("));
```

$$.5983922646$$

The next example illustrates that sometimes the element of volume can be hollow.

Example 8.2.4 Find the volume of the solid obtained by rotating the region bounded by the curves $y = 2\sin x$ and $y = x$ about the x-axes.

Solution: First plot the region bounded between the two curves.

```
> f := x -> 2*sin(x); g := x -> x;
```

$$f := x \mapsto 2\sin(x)$$

$$g := x \mapsto x$$

The following Maple V segment plots the region bounded between the curves along with a typical element of area. See Figure 37.

```
> rect := plots[polygonplot]([[0.9,g(1)],[1.1,g(1)]
> ,[1.1,f(1)],[0.9,f(1)]]):
> plt := plot({f(x),g(x)},x=0..2):
> plots[display]({plt,rect});
```

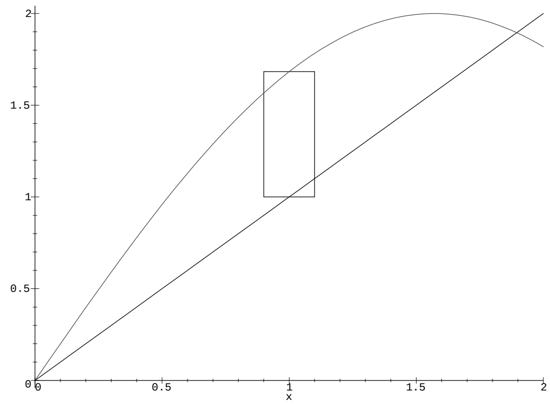

Figure 37: Region bounded by $y = 2\sin x$, and $y = x$.

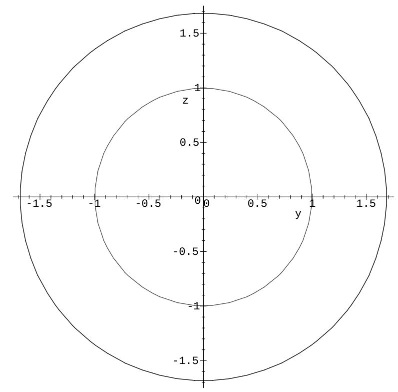

Figure 38: Cross-sectional area of an element of volume

Now rotate the element of area about the x-axis to create a cylindrical solid with a hole drilled through the center. Figure 38 is a cross-sectional view of this cylinder as viewed looking down the positive x-axis.

```
> plots[implicitplot]({y^2+z^2=g(1)^2,y^2+z^2=f(1)^2},
> y=-2..2,z=-2..2, scaling=constrained);
```

Note that the area of a typical cross-section is equal to the area of the larger circle minus the area of the smaller circle:

$$\Delta A = \pi \cdot f(x)^2 - \pi \cdot g(x)^2.$$

This means that an element of volume is

$$\Delta V = \pi \cdot (f(x)^2 - g(x)^2) \cdot \Delta x.$$

Now try to imagine the solid formed by revolving the entire region about the x-axis. Figure 39 is a point plot of the solid.

```
> plt1 := plots[tubeplot]([x,0,0],x=0..2,radius=f(x)):
> plt2 := plots[tubeplot]([x,0,0],x=0..2,radius=g(x)):
> plots[display]({plt1,plt2},axes=normal,style=point);
```

The volume of this solid of revolution is equal to

$$V = \pi \int_a^b (f(x)^2 - g(x)^2)\, dx,$$

where a and b are the x coordinates of the points of intersection of the two curves. It is easy to verify that $a = 0 = \sin 0$. Use **fsolve** to approximate b to ten digits of accuracy.

```
> b := fsolve(f(x)=g(x),x,0.5..2);
                          b := 1.895494267
```

The integral is calculated.

```
> Int(Pi*(f(x)^2-g(x)^2),x=0..b): "=value(");
```

$$\int_0^{1.895494267} \pi\left(4\,(\sin(x))^2 - x^2\right) dx = 6.677730766$$

Therefore, the volume of the solid is approximately 6.677730766.

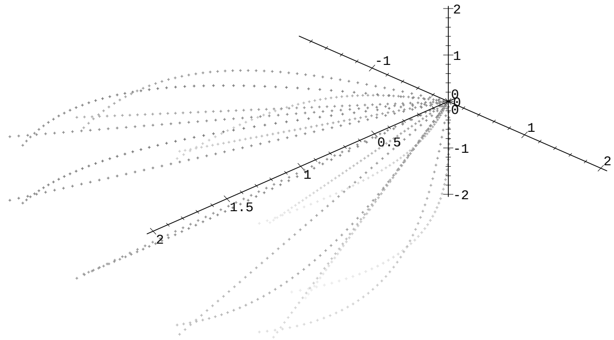

Figure 39: Region revolve around x-axis

Arc Length

The definite integral can also be used to compute the length of a smooth curve. Recall that when using the integral to find the area of a region one approximates the region by rectangles the sum of whose areas approximate the area of the region. In finding the length of an arc one approximates the arc by a finite set of straight line segments. An approximation of the length of the arc is made by using the well known formula for the length of a line segment and taking a sum . A limiting process then yields the definite integral which is equal to the length of the arc.

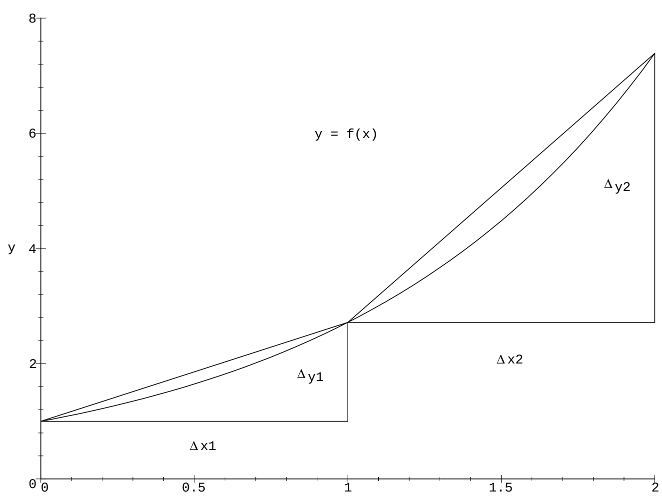

Figure 40: The curve $y = f(x)$ approximated by two line segments

For example in Figure 40 the curve $y = f(x)$ is approximated by two line segments with slope

$$\frac{\Delta y_1}{\Delta x_1} = \frac{f(1) - f(0)}{1 - 0}$$

and

$$\frac{\Delta y_2}{\Delta x_2} = \frac{f(2) - f(1)}{2-1},$$

respectively. The length of the two arcs is equal to

$$\sqrt{\Delta x_1^2 + \Delta y_1^2} = \sqrt{1 + \left(\frac{\Delta y_1}{\Delta x_1}\right)^2}\ \Delta x_1$$

and

$$\sqrt{\Delta x_2^2 + \Delta y_2^2} = \sqrt{1 + \left(\frac{\Delta y_2}{\Delta x_2}\right)^2}\ \Delta x_2,$$

respectively. The sum of these two lengths is

$$\sum_{i=1}^{2} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\ \Delta x_i.$$

This suggests a Riemann Sum with summand

$$\sqrt{1 + f'(c_i)^2}\,\Delta x_i,$$

since, if f is differentiable, the mean value theorem implies

$$f'(c_i) \approx \frac{\Delta y_i}{\Delta x_i}.$$

Taking a limit of Riemann Sums of this type as Δx_i tends to zero, one has the following formula for the length of a smooth curve given by $y = f(x)$, with f a differentiable function defined on a closed interval $[a, b]$.

$$\boxed{\text{Arc length } = L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.}$$

Example 8.2.5 Find the length of the arc given by the equation $y^2 = x^3$ between the points $(1, 1)$ and $(5, \sqrt[3]{125})$.

Solution: The curve which is given by the equation

```
> eq := y^2=x^3;
```

$$eq := y^2 = x^3$$

is shown in Figure 41, where **implicitplot** has been used.

```
> plots[implicitplot](eq,x=0..12,y= 0..12);
```

In order to apply the formula for arc length we must define a function whose graph is the arc. This can be done with **solve**.

```
> solve(eq,y);
```

$$x^{3/2}, -x^{3/2}$$

The function that defines the arc must be equal to 1 when $x = 1$, so choose the positive expression from the previous Maple V output.

```
> f := "[1];
```

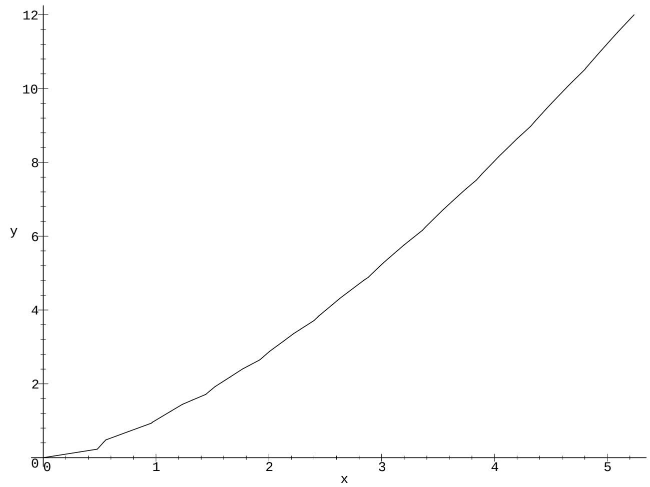

Figure 41: The curve $y^2 = x^2$

$$f := x^{3/2}$$

Convert f to a function using **unapply**.

```
> f := unapply(f,x);
```

$$f := x \mapsto x^{3/2}$$

The formula for arc length requires f'.

```
> fprime := D(f);
```

$$fprime := x \mapsto \frac{3\sqrt{x}}{2}$$

Compute the length of the arc between $(1,1)$, and $(5, \sqrt[3]{125})$ by the following.

```
> Int(sqrt(1+fprime(x)^2),x=1..5): " = value(");
```

$$\int_1^5 \frac{\sqrt{4+9x}}{2} dx = \frac{343}{27} - \frac{13\sqrt{13}}{27}$$

This is approximately equal to

```
> evalf(rhs("));
```

10.96769753

Example 8.2.6 Find the length of the perimeter of the ellipse

$$\frac{x^2}{4} + \frac{y^2}{25} = 1.$$

Solution: The ellipse given by the equation

```
> eq := x^2/4+y^2/25=1;
```

$$eq := \frac{x^2}{4} + \frac{y^2}{25} = 1$$

is shown in Figure 42.

```
> plots[implicitplot](eq,x=-2..2,y=-5..5,scaling=constrained);
```

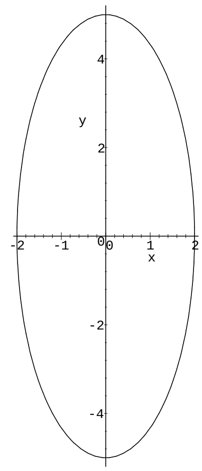

Figure 42: The curve $\frac{x^2}{4} + \frac{y^2}{25} = 1$

It is clear from Figure 42 that the length of the perimeter is equal to four times the length of the arc of the ellipse from the point $(0, 5)$ to $(2, 0)$. Now define a function f such that $y = f(x)$ gives this arc.

```
> solve(eq,y);
```

$$\frac{5\sqrt{-x^2+4}}{2}, -\frac{5\sqrt{-x^2+4}}{2}$$

```
> f := "[1];
```

$$f := \frac{5\sqrt{-x^2+4}}{2}$$

Next the expression is converted into a function.

```
> f := unapply(f,x);
```

$$f := x \mapsto \frac{5\sqrt{-x^2+4}}{2}$$

The derivative f' is needed.

```
> fprime := D(f);
```

$$fprime := x \mapsto -\frac{5\,x}{2\sqrt{-x^2+4}}$$

We now compute the length of the perimeter of the ellipse.

```
> 4*Int(sqrt(1+fprime(x)^2),x=0..2): " := evalf(");
```

```
" := 23.01311260
```

Exercises 8.2

1. Show that the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is πab. Hint: Consider using **assume**(a>0); and **additionally**(a<1);

2. Plot the figure eight
$$(1+x^2)\cdot y^2 = x^2\cdot(1-x^2),$$
and then show that the enclosed area is equal to $\pi - 2$.

3. Consider the region bounded by the curve $y = x\sin^2 x$, between $x = 0$, $x = \pi$, and the x-axis. Plot this region and then calculate the exact value of the volume of the solid formed by revolving the region about the x-axis.

4. Compute the volume of the solid of revolution which is formed by revolving the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
about the $x - axis$.

5. Plot the curve
$$y = \sin(\pi \sin((x-2)^2))$$
over the interval $[3, 5]$. Estimate the arc length of the curve between $x = 3$ and $x = 5$.

6. If $a > 0$ and $b > 0$ then show that the arc length of the curve
$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$$
is
$$\frac{a^2+ab+b^2}{a+b}.$$
Plot the curve.

7. Let $f(x) = \cosh(x)$ and $g(x) = 2\cosh(x)$. (Recall: $\cosh(x) = \frac{e^x+e^{-x}}{2}$.)
   (a) Find the area bounded by the curve $y = f(x)$ and the x-axis from $x = 0$ to $x = 1$. Do the same for $y = g(x)$, and then compute the ratio of the two areas.
   (b) Find the volumes of the solids obtained by rotating each of the regions in part (a) about the x-axis. Compute the ratio of the two volumes.
   (c) Find the lengths of the two curves, $y = f(x)$ and $y = g(x)$, from $x = 0$ to $x = 1$, and calculate the ratios of te two lengths.

8. In this problem we will study what happens to the ratio of arc length to area as $a \to \infty$ for three curves that depend on a parameter a. For each of the three functions given below perform the following tasks:
   (a) Plot the graph of the function for $a = 1$.
   (b) With $a = 1$, find the area bounded by the curve $y = f(x)$ and the x-axis on $[0, 1]$.
   (c) With a pencil and paper write down integral formulas for the arc length of the curve $y = f(x)$ over the interval $[0, 1]$, and the area under the curve on $[0, 1]$. Use these to find an integral formula for the limit of the ratio of arc length to area as $a \to \infty$. (Hint: Factor out a from each integral before taking the limit.)
   (d) Find the limit as $a \to \infty$ of the ratio of arclength to area on $[0, 1]$.
   (e) By studying the graph found in (a), and using the answer you calculated in (b) can you find a way to answer (d) without doing any further integrations?
      i. $f(x) = a\sin^2(\pi x)$
      ii. $f(x) = a\sqrt{x - x^2}$ (Hint: You may wish to use only geometry here.)
      iii. $f(x) = x\ H(1/2 - x) + (1-x)\ H(x - 1/2)$, where H is the Heaviside function.

8.3 Applications to Physics

It has been seen how calculus can be applied to find solutions of geometric problems such as problems concerned with computing area, volume, and arc length. In this section calculus is used to solve problems that arise from physics.

Work

Consider problems which involve the physical concept of work. The definition of the work done by a constant force, F, in moving an object a distance, d, is equal to the product of the force and the distance moved.

Definition
When a body moves a distance d along a straight line under the action of a constant force F in the direction of the motion, then the *work* W done by the force in moving the object is

$$\text{Work} = \text{Force} \times \text{Distance}$$

or

$$W = F \cdot d$$

In most cases the applied force is not constant, but varies over the straight line. For example suppose that the force, $F(x)$, acting on a particle as it moves along the straight line from a to b varies continuously. See Figure 43.

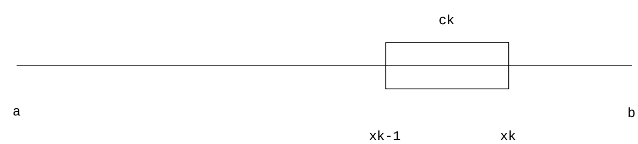

Figure 43: Force along an interval

Consider the particle's movement over a very small subinterval $[x_{k-1}, x_k]$ of the original interval. Then, since F is continuous, one can approximate its value over a sufficiently small subinterval by taking its value at an arbitrary point, c_k, in in that interval. Then the work done in moving the particle from x_{k-1} to x_k is approximately

$$F(c_k) \cdot (x_k - x_{k-1}) = F(c_k) \cdot \Delta x_k.$$

To obtain an approximation to the total work done add up all of the similar *elements of work.*

$$\text{Work} \quad \approx \sum F(c_k) \cdot \Delta x_k.$$

This represents a Riemann Sum of the continuous function $F(x)$ over the interval $[a, b]$. Just as the area of a planar region described by continuous curves can defined as a limit of Riemann Sums, the work done by a continuous force acting to move a particle between two points on a line may be defined as a limit of Riemann Sums, or what is the same thing the definite interval.

Definition
The **work** done by a continuous force $F(x)$ directed along the x-axis is

$$W = \int_a^b F(x)\, dx.$$

Example 8.3.1 Suppose that you wish to draw water from a well which has a water level located 20 feet below the mouth of the well. Your 2 gallon bucket weighs 4 lbs, and the rope weighs 0.10 lb/ft. Unfortunately, your bucket has a leak and even though it is originally full, it has only a gallon of water by the time you lift it to the top. Assuming that you pull the bucket up at a constant rate and the water leaks at a constant rate determine the work you do in lifting the bucket of water to the top. Assume that water weighs 8 lbs per gallon.

Solution: Consider a linear coordinate system with the origin at the mouth of the well. The coordinate at the the mouth is $x = 0$, and the coordinate at the water level is $x = 20$. The force, $F(x)$, that is required to lift the water $F_w(x)$, the bucket, $F_b(x)$, and the rope $F_r(x)$ will now be computed with

$$F(x) = F_w(x) + F_b(x) + F_r(x).$$

1. The force contributed by the bucket is a constant, since its weight at any depth is always 4 lbs. Thus

$$F_b(x) = 4 \text{ lbs.}$$

2. The force contributed by the rope varies with the depth. When 20 feet of rope is out the total weight is $0.10 \text{ lb/ft } \times 20 = 2$ lbs, and at the mouth the rope is of 0 length and hence weighs 0 lbs. The weight of the rope at a point x ft below the mouth is

$$F_r(x) = 0.10 \times x = \frac{x}{10} \text{ lbs.}$$

3. Since the bucket leaks the weight of the water varies with depth. When the bucket starts its ascent it contains two gallons of water which weighs 16 lbs, and the bucket is half full at the top and weighs 8 lbs. It is assumed that the bucket moves up at a constant rate and the water is leaking out at a constant rate. A formula for the weight of water at a depth x can be determined. When the rope is fully extended to 20 ft the weight of the water is 16 lbs. When the bucket is raised to the top the weight of the water is 8 lbs. The bucket is raised at a constant rate, say v ft/sec and the bucket leaks at a constant rate, say k lbs/sec. The time, τ, that it takes to raise the bucket 20 ft is the same as the time for 8 lbs of water to leak out. Thus

$$\tau = \frac{8}{k} = \frac{20}{v}.$$

This means that

$$\frac{k}{v} = \frac{8}{20} = \frac{2}{5}.$$

Now the weight of water remaining after time t is

$$\text{Weight } = 16 - k \cdot t$$

and the length of rope that is out at time t is

$$x = 20 - v \cdot t.$$

Solving the latter equation for t, substituting the result into the equation for weight, and using the fact that $\frac{k}{v} = \frac{2}{5}$ gives:

$$\text{Weight } = 16 - \frac{k}{v} \cdot (20 - x) = 8 + \frac{2}{5}x.$$

The last equation defines the weight of the water in the bucket at a depth x. Hence we write

$$F_w = 8 + \frac{2}{5}x.$$

The force required to lift bucket, rope, and water from a depth of x feet is

$$F(x) = (4) + (\frac{x}{10}) + (8 + \frac{2}{5}x) = 12 + \frac{x}{2}.$$

The amount of work done in lifting the bucket, rope, and water the 20 ft

$$W = \int_0^{20} (12 + \frac{x}{2})\, dx = 12x + \frac{x^2}{4}\Big|_0^{20} = 240 + 100 = 340 \text{ ft} - \text{lbs}.$$

When a body is near the surface of the Earth it is usually assumed that the force acting on it due to gravity is constant *i.e.* the object's weight as in the preceding example. However, as the body moves further from the Earth's surface the force acting to bring it back to earth varies inversely as the square of the distance from the center of the earth.

Newton's Law of Gravity
Let m_1, and m_2 be the masses of two objects that are a distance r apart. The force, F, of attraction due gravity is

$$F = \frac{Gm_1m_2}{r^2},$$

where G is the *universal gravitational constant* and depends upon the units of distance, mass, force, and time that are used. If these units are meters (m), kilograms (kg), Newtons, (N) and seconds, then $G = 6.6720 \times 10^{-11}$ Nm^{2}kg^{-2}.

Example 8.3.2 Find the work required to move a satellite of mass 1250 kg from the surface of the Earth (of mass 5.975×10^{24} kg), to the surface of the Moon (of mass 7.35×10^{22} kg). Assume that the radius of the earth, ER, is 6.38×10^6 meters, the moon's radius, MR, is 1.74×10^6 meters, and that the distance, DEM, from the center of the Earth to the center of the Moon is 3.84×10^8 meters.

Solution: We let SM, EM, and MM denote the masses of the satellite, Earth, respectively.

```
> SM:= 1250;  EM := 5.975*10^24;  MM := 7.35*10^22;
```

$$SM := 1250$$

$$EM := 5.975000000 \times 10^{24}$$

$$MM := 7.350000000 \times 10^{22}$$

```
> G := 6.6720*10^(-11); DEM := 3.84*10^8;
```

$$G := 6.672010^{\{-11\}}$$

$$DEM := 384000000.0$$

```
> ER := 6.38*10^6;   MR := 1.74*10^6;
```

$$ER := 6380000.0$$

$$MR := 1740000.0$$

Let r denote the distance that the satellite is from the center of the Earth, then the force, $FE(r)$, acting to pull it back to Earth is

$$\frac{G \cdot EM \cdot SM}{r^2}.$$

```
> FE := r -> G*SM*EM/(r^2);
```

$$r \mapsto \frac{498315000000000000.0}{r^2}$$

At the same time the Moon is attracting the satellite with force, $FM(r)$ is

$$\frac{G \cdot MM \cdot SM}{(DEM - r)^2}.$$

```
> FM := r -> G*SM*MM/(DEM-r)^2;
```

$$FM := \frac{6129900000000000.0}{(384000000.0 - r)^2}$$

Thus the entire force $F(r)$ acting on the satellite is

$$F(r) = \frac{G \cdot EM \cdot SM}{r^2} - \frac{G \cdot MM \cdot SM}{(DEM - r)^2}.$$

This means that the work required to move the satellite from the surface of the earth is given by

$$\int_{ER}^{MR} F(r)\, dr.$$

```
> Int(FE(r)-FM(r),r=ER..(DEM-MR)): " = evalf(");
```

$$\int_{6380000.0}^{382260000.0} \frac{498315000000000000.0}{r^2} - \frac{6129900000000000.0}{(384000000.0 - r)^2} dr = 73079043410.0$$

The work required to move the satellite from the surface of the Earth to the surface of the Moon is $73, 079, 043, 410$ Newton-meters or joules of work.

Example 8.3.3 A tank in the shape of a one foot high frustrum of a cone has a base radius of two feet, and a radius at the top of three feet, is filled with a liquid which weighs 65 pounds per cubic feet. See Figure 44. How much work is required to pump all of the liquid to a height of two feet above the frustrum.

Solution: Choose coordinates (x, y, z) so that z is height measured from bottom of the tank. Consider a thin cylindrical disk, ΔV, as an element of volume for the cone (the same as if you were going to find the volume of the frustrum). The typical base for such a disk located at a height z has a radius x equal to

$$x = 1 + 2z,$$

and height Δz. Thus

$$\Delta V = \pi(1 + 2z)^2 \Delta z$$

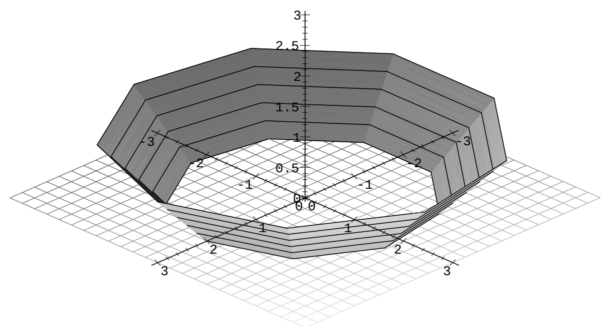

Figure 44: Tank containing a liquid

cubic feet. If the element of volume is filled with the liquid then it would weigh $65\,\pi(z-2)^2\Delta z$. To lift this much liquid to a height two feet above the top would require

$$\Delta w = (3-z) \times 65 \times \pi(1+2z)^2 \Delta z$$

foot-pounds since the liquid must be lifted $3-z$ feet. This means that the total work is a limit of sums of these elements and hence the work required to lift all of the liquid to a height 2 feet above the top is equal to

$$w = \int_0^1 (3-z) \times 65 \times \pi(1+2z)^2 \, dx$$

```
> Int(Pi*65*(3-z)*(1+2*z)^2,z=0..1): " = value(");
```

$$\int_0^1 65\,\pi\,(3-z)\,(1+2\,z)^2\,dz = \frac{3965\,\pi}{6}$$

```
> evalf(rhs("));
```

2076.069145

Thus it requires approximately 2076.069145 foot-pounds of work to pump all of the liquid to to a height two feet above the top of the frustrum.

Exercises 8.3

1. The bucket in Example 8.3.1 is pulled faster and there is a gallon and a half remaining when it reaches the top. What is the work done in this case?

2. Using the physical constants from Example 8.3.2 calculate the amount of work necessary to to carry a rocket with mass $3,238$ kg from the Earth's surface to a height $43,257$ meters above the Earth's surface.

3. A fluid which weighs 65 pounds per cubic feet is contained in a well in the shape of a sphere of radius 67 feet. The top of the sphere is 56 feet below the surface of the Earth. How much work is required to pump all of the fluid up to ground level?